ℚ-FANO THREEFOLDS OF LARGE FANO INDEX, I

YURI PROKHOROV

ABSTRACT. We study ℚ-Fano threefolds of large Fano index. In particular, we prove that the maximum of Fano index is attained for the weighted projective space $\mathbb{P}(3,4,5,7)$.

1. INTRODUCTION

The Fano index of a smooth Fano variety X is the maximal integer $\mathrm{q}(X)$ that divides the anti-canonical class in the Picard group $\mathrm{Pic}(X)$ [IP99]. It is well-known [KO73] that $\mathrm{q}(X) \leq \dim X + 1$. Moreover, $\mathrm{q}(X) = \dim X + 1$ if and only if X is a projective space and $\mathrm{q}(X) = \dim X$ if and only if X is a quadric hypersurface. In this paper we consider generalizations of Fano index for the case of singular Fanos admitting terminal singularities.

A normal projective variety X is called *Fano* if some positive multiple $-nK_X$ of its anti-canonical Weil divisor is Cartier and ample. Such X is called a ℚ-*Fano variety* if it has only terminal ℚ-factorial singularities and its Picard number is one. This class of Fano varieties is important because they appear naturally in the Minimal Model Program.

For a singular Fano variety X the Fano index can be defined in different ways. For example, we can define

$$\mathrm{qW}(X) \quad := \quad \max\{q \mid -K_X \sim qA, \qquad A \text{ is a Weil } \mathbb{Q}\text{-Cartier divisor}\},$$

$$\mathrm{q}\mathbb{Q}(X) \quad := \quad \max\{q \mid -K_X \sim_{\mathbb{Q}} qA, \qquad \underline{\qquad\qquad " \qquad\qquad} \quad \}.$$

If X has at worst log terminal singularities, then the Picard group $\mathrm{Pic}(X)$ and Weil divisor class group $\mathrm{Cl}(X)$ are finitely generated and $\mathrm{Pic}(X)$ is torsion free (see e.g. [IP99, §2.1]). Moreover, the numerical equivalence of ℚ-Cartier divisors coincides with ℚ-linear one. This implies, in particular, that defined above Fano indices $\mathrm{qW}(X)$ and $\mathrm{q}\mathbb{Q}(X)$ are positive integers. If X is smooth, these numbers coincide with the Fano index $\mathrm{q}(X)$ defined above. Note also that $\mathrm{q}\mathbb{Q}(X) = \mathrm{qW}(X)$ if the group $\mathrm{Cl}(X)$ is torsion free.

Theorem 1.1 ([Suz04]). *Let X be a ℚ-Fano threefold. Then $\mathrm{qW}(X) \in \{1, \ldots, 11, 13, 17, 19\}$. All these values, except possibly for $\mathrm{qW}(X) = 10$,*

occur. Moreover, if $\mathrm{qW}(X) = 19$*, then the types of non-Gorenstein points and Hilbert series of* X *coincide with that of* $\mathbb{P}(3, 4, 5, 7)$*.*

It can be easily shown (see proof of Proposition 3.6) that the index $\mathrm{q}\mathbb{Q}(X)$ takes values in the same set $\{1, \ldots, 11, 13, 17, 19\}$. Thus one can expect that $\mathbb{P}(3, 4, 5, 7)$ is the only example of $\mathbb{Q}$-Fano threefols with $\mathrm{q}\mathbb{Q}(X) = 19$. In general, we expect that Fano varieties with extremal properties (maximal degree, maximal Fano index, etc.) are quasihomogeneous with respect to an action of some connected algebraic group. This is supported, for example, by the following facts:

Theorem 1.2 ([Pro05], [Pro07])**.**   (i) *Let* X *be a* $\mathbb{Q}$*-Fano threefold. Assume that* X *is not Gorenstein. Then* $-K_X^3 \leq 125/2$ *and the equality holds if and only if* X *is isomorphic to the weighted projective space* $\mathbb{P}(1^3, 2)$*.*

   (ii) *Let* X *be a Fano threefold with canonical Gorenstein singularities. Then* $-K_X^3 \leq 72$ *and the equality holds if and only if* X *is isomorphic to* $\mathbb{P}(1^3, 3)$ *or* $\mathbb{P}(1^2, 6, 4)$*.*

The following proposition is well-known (see, e.g., [BB92]). It is an easy exercise for experts in toric geometry.

Proposition 1.3. *Let* X *be a toric* $\mathbb{Q}$*-Fano 3-fold. Then* X *is isomorphic to either* $\mathbb{P}^3$*,* $\mathbb{P}^3/\boldsymbol{\mu}_5(1, 2, 3)$*, or one of the following weighted projective spaces:*

$\mathbb{P}(1^3, 2)$*,* $\mathbb{P}(1^2, 2, 3)$*,* $\mathbb{P}(1, 2, 3, 5)$*,* $\mathbb{P}(1, 3, 4, 5)$*,* $\mathbb{P}(2, 3, 5, 7)$*,* $\mathbb{P}(3, 4, 5, 7)$*.*

We characterize the weighted projective spaces above in terms of Fano index. The following is the main result of this paper.

Theorem 1.4. *Let* X *be a* $\mathbb{Q}$*-Fano threefold. Then* $\mathrm{q}\mathbb{Q}(X) \in \{1, \ldots, 11, 13, 17, 19\}$*.*

  (i) *If* $\mathrm{q}\mathbb{Q}(X) = 19$*, then* $X \simeq \mathbb{P}(3, 4, 5, 7)$*.*
  (ii) *If* $\mathrm{q}\mathbb{Q}(X) = 17$*, then* $X \simeq \mathbb{P}(2, 3, 5, 7)$*.*
  (iii) *If* $\mathrm{q}\mathbb{Q}(X) = 13$ *and* $\dim |-K_X| > 5$*, then* $X \simeq \mathbb{P}(1, 3, 4, 5)$*.*
  (iv) *If* $\mathrm{q}\mathbb{Q}(X) = 11$ *and* $\dim |-K_X| > 10$*, then* $X \simeq \mathbb{P}(1, 2, 3, 5)$*.*
  (v) $\mathrm{q}\mathbb{Q}(X) \neq 10$*.*
  (vi) *If* $\mathrm{q}\mathbb{Q}(X) \geq 7$ *and there are two effective Weil divisors* $A \neq A_1$ *such that* $-K_X \sim_{\mathbb{Q}} \mathrm{q}\mathbb{Q}(X)A \sim_{\mathbb{Q}} \mathrm{q}\mathbb{Q}(X)A_1$*, then* $X \simeq \mathbb{P}(1^2, 2, 3)$*.*
  (vii) *If* $\mathrm{qW}(X) = 5$ *and* $\dim |-\frac{1}{5}K_X| > 1$*, then* $X \simeq \mathbb{P}(1^3, 2)$*.*

Note that in cases (iii) and (iv) assumptions about $\dim |-K_X|$ are needed. Indeed, there are examples of non-toric $\mathbb{Q}$-Fano threefolds with $\mathrm{q}\mathbb{Q}(X) = 13$ and 11.

In the proof we follow the use some techniques developed in our previous paper [Pro07]. By Proposition 1.3 it is sufficient to show that our $\mathbb{Q}$-Fano X is toric. First, as in [Suz04], we apply the orbifold Riemann-Roch

formula to find all the possibilities for the numerical invariants of X. In all cases there is some special element $S \in |-K_X|$ having four irreducible components. This S should be a toric boundary, if X is toric. Further, we use birational transformations like Fano-Iskovskikh "double projection" [IP99] (see [Ale94] for the $\mathbb{Q}$-Fano version). Typically the resulting variety is a Fano-Mori fiber space having "simpler" structure. (In particular, its Fano index is large if this variety is a $\mathbb{Q}$-Fano). By using properties of our "double projection" we can show that the pair (X, S) is log canonical (LC). Then, in principle, the assertion follows by Shokurov's toric conjecture [McK01]. We prefer to propose an alternative, more explicit proof. In fact, the image of X under "double projection" is a toric variety and the inverse map preserves the toric structure. In the last section we describe Sarkisov links between toric $\mathbb{Q}$-Fanos that starts with blowing ups *singular* points.

2. Preliminaries, the orbifold Riemann-Roch formula and its applications

Notation. Throughout this paper, we work over the complex number field $\mathbb{C}$. We employ the following standard notation:

$\sim$ denotes the linear equivalence;

$\sim_{\mathbb{Q}}$ denotes the $\mathbb{Q}$-linear equivalence.

Let E be a rank one discrete valuation of the function field $\mathbb{C}(X)$ and let D is a $\mathbb{Q}$-Cartier divisor on X. $a(E, D)$ denotes the discrepancy of E with respect to a boundary D. Let $f \colon \tilde{X} \to X$ be a birational morphism such that E appears as a prime divisor on $\tilde{X}$. Then $\mathrm{ord}_E(D)$ denotes the coefficient of E in f^*D.

2.1. The orbifold Riemann-Roch formula [Rei87]**.** Let X be a three-fold with terminal singularities and let D be a Weil $\mathbb{Q}$-Cartier divisor on X. Let $\mathbf{B} = \{(r_P, b_P)\}$ be the basket of singular points of X [Mor85a], [Rei87]. Here each pair (r_P, b_P) correspond to a point $P \in \mathbf{B}$ of type $\frac{1}{r_P}(1, -1, b_P)$. For brevity, describing a basket we will list just indices of singularities, i.e., we will write $\mathbf{B} = \{r_P\}$ instead of $\mathbf{B} = \{(r_P, b_P)\}$. In the above situation,

the Riemann-Roch formula has the following form

$$(2.2) \quad \chi(D) = \frac{1}{12} D \cdot (D - K_X) \cdot (2D - K_X) +$$

$$+ \frac{1}{12} D \cdot c_2 + \sum_{P \in \mathbf{B}} c_P(D) + \chi(\mathcal{O}_X),$$

where

$$c_P(D) = -i_P \frac{r_P^2 - 1}{12 r_P} + \sum_{j=1}^{i_P - 1} \frac{\overline{b_P j}(r_P - \overline{b_P j})}{2 r_P}.$$

Clearly, computing $c_P(D)$, we always may assume that $1 \le b_P \le r_P/2$.

2.3. Now let X be a Fano threefold with terminal singularities, let $q := \mathrm{q}\mathbb{Q}(X)$, and let A be an ample Weil $\mathbb{Q}$-Cartier divisor on X such that $-K_X \sim_{\mathbb{Q}} qA$. By (2.2) we have

$$(2.4) \quad \chi(tA) = 1 + \frac{t(q+t)(q+2t)}{12} A^3 + \frac{tA \cdot c_2}{12} + \sum_{P \in \mathbf{B}} c_P(tA),$$

$$c_P(tA) = -i_{P,t} \frac{r_P^2 - 1}{12 r_P} + \sum_{j=1}^{i_{P,t} - 1} \frac{\overline{b_P j}(r_P - \overline{b_P j})}{2 r_P}.$$

If $q > 2$, then $\chi(-A) = 0$. Using this equality we obtain (see [Suz04])

$$(2.5) \quad A^3 = \frac{12}{(q-1)(q-2)} \left(1 - \frac{A \cdot c_2}{12} + \sum_{P \in B} c_P(-A) \right).$$

In the above notation, applying (2.2), Serre duality and Kawamata-Viehweg vanishing to $D = K_X$, we get the following important equality (see, e.g., [Rei87]):

$$(2.6) \quad 24 = -K_X \cdot c_2(X) + \sum_{P \in \mathbf{B}} \left(r_P - \frac{1}{r_P} \right).$$

Theorem 2.7 ([Kaw92a], [KMMT00]). *In the above notation,*

$$(2.8) \quad -K_X \cdot c_2(X) \ge 0, \qquad \sum_{P \in \mathbf{B}} \left(r_P - \frac{1}{r_P} \right) \le 24.$$

Proposition 2.9. *Let X be a Fano thereefold with terminal singularities and let Ξ be an n-torsion element in the Weil divisor class group. Let $\mathbf{B}^{\Xi}$ be the collection of points $P \in \mathbf{B}$ where Ξ is not Cartier. Then*

$$(2.10) \quad 2 = \sum_{P \in \mathbf{B}^{\Xi}} \frac{\overline{b_P i_{\Xi,P}} \left(r_P - \overline{b_P i_{\Xi,P}} \right)}{2 r_P}.$$

where $i_{\Xi,P}$ is taken so that $\Xi \sim i_{\Xi,P}K_X$ near $P \in \mathbf{B}$ and $\overline{}$ is the residue mod r_P. Assume furthermore that n is prime. Then

    (i) $n \in \{2,3,5,7\}$.
    (ii) If $n = 7$, then $\mathbf{B}^{\Xi} = (7,7,7)$.*
    (iii) If $n = 5$, then $\mathbf{B}^{\Xi} = (5,5,5,5)$, $(10,5,5)$, or $(10,10)$.
    (iv) If $n = 3$, then $\sum_{P \in \mathbf{B}^{\Xi}} r_P = 18$.
    (v) If $n = 2$, then $\sum_{P \in \mathbf{B}^{\Xi}} r_P = 16$.

Proof. Let By Riemann-Roch (2.2), Kawamata-Viehweg vanishing theorem and Serre duality we have

$$0 \;\; = \chi(\Xi) \qquad\quad = 1 + \sum_P c_P(\Xi),$$

$$0 \;\; = \chi(K_X + \Xi) \;\; = 1 + \tfrac{1}{12}K_X \cdot c_2(X) + \sum_{P \in \mathbf{B}} c_P(K_X + \Xi).$$

Subtracting we get

$$0 = -\frac{1}{12}K_X \cdot c_2(X) + \sum_{P \in \mathbf{B}}(c_P(\Xi) - c_P(K_X + \Xi)).$$

Since $ni_{\Xi,P} \equiv 0 \mod r_P$,

$$0 = -\frac{1}{12}K_X \cdot c_2(X) + \frac{1}{12}\sum_{P \in \mathbf{B}}\left(r_P - \frac{1}{r_P}\right) - \sum_{P \in \mathbf{B}}\frac{\overline{b_P i_{\Xi,P}}\left(r_P - \overline{b_P i_{\Xi,P}}\right)}{2r_P}.$$

This proves (2.10).

Now assume that n is prime. If $P \in \mathbf{B}^{\Xi}$, then $n \mid r_P$. Write $r_P = nr'_P$. Since $r_P \mid ni_P$, $i_P = r'_P i'_P$, where $n \nmid i'_P$. Let $\overline{(\;\;)}_n$ be the residue mod n. Then

$$2 = \sum_{P \in \mathbf{B}^{\Xi}}\frac{\overline{b_P i'_{\Xi,P} r'}\left(nr'_P - \overline{b_P i'_{\Xi,P} r'_P}\right)}{2nr'_P} = \frac{r'_P \overline{(b_P i'_{\Xi,P})_n}\left(n - \overline{(b_P i'_{\Xi,P})_n}\right)}{2n}.$$

Therefore,

$$4n^2 = \sum_{P \in \mathbf{B}^{\Xi}} r_P \overline{(b_P i'_{\Xi,P})_n}\left(n - \overline{(b_P i'_{\Xi,P})_n}\right).$$

Denote $\xi_P := \overline{(b_P i'_{\Xi,P})_n}$. Then $0 < \xi_P < n$, $\gcd(n,\xi_P) = 1$, and

$$4n = \sum_{P \in \mathbf{B}^{\Xi}} r'_P \xi_P (n - \xi_P) \geq \frac{n^2}{4}\sum_{P \in \mathbf{B}^{\Xi}} r'_P, \qquad 16 \geq n \sum_{P \in \mathbf{B}^{\Xi}} r'_P.$$

If $n \geq 11$, then $\sum r'_P = 1$, $n \mid r'_P$, and $r_P \geq n^2 \geq 121$, a contradiction. Therefore, $n \leq 7$. Consider the case $n = 7$. Then $\xi_P (n - \xi_P) = 6, 10$, or 12. The only solution is $\mathbf{B}^{\Xi} = (7,7,7)$. The case $n = 5$ is considered similarly. If $n = 3$, then $\xi_P (n - \xi_P) = 3$ and $\sum r_P = 3\sum r'_P = 18$. Similarly, if

---

    *More delicate computations show that this case does not occur. (We do not need this.)

$n = 2$, then $\xi_P (n - \xi_P) = 1$ and $\sum r_P = 2 \sum r'_P = 16$. This finishes the proof. $\qquad\square$

3. Computations with Riemann-Roch on $\mathbb{Q}$-Fano threefolds of large Fano index

Lemma 3.1 (see [Suz04])**.** *Let X be a Fano threefold with terminal singularities with $q := \mathrm{qW}(X)$, let $A := -\frac{1}{q}K_X$, and let r be the Gorenstein index of X. Then*

    (i) *r and q are coprime;*
    (ii) *rA^3 is an integer.*

Lemma 3.2. *Let X be a Fano threefold with terminal singularities.*

    (i) *If $-K_X \sim qL$ for some Weil divisor L, then q divides $\mathrm{qW}(X)$.*
    (ii) *If $-K_X \sim_{\mathbb{Q}} qL$ for some Weil divisor L, then q divides $\mathrm{q\mathbb{Q}}(X)$.*
    (iii) *$\mathrm{qW}(X)$ divides $\mathrm{q\mathbb{Q}}(X)$.*
    (iv) *Let $q := \mathrm{q\mathbb{Q}}(X)$ and let $K_X + qA \sim_{\mathbb{Q}} 0$. If the order of $K_X + qA$ in the group $\mathrm{Cl}(X)$ is prime to q, then $\mathrm{qW}(X) = \mathrm{q\mathbb{Q}}(X)$.*

Proof. To prove (i) write $-K_X \sim \mathrm{qW}(X)A$ and let $d = \gcd(\mathrm{qW}(X), q)$. Then $d = u\,\mathrm{qW}(X) + vq$ for some $u, v \in \mathbb{Z}$. Hence, $dA = u\,\mathrm{qW}(X)A + vqA \sim quL + qvA = q(uL + vA)$. Since A is a primitive element of $\mathrm{Cl}(X)$, $q = d$ and $q \mid \mathrm{qW}(X)$.

(ii) can proved similarly and (iii) is a consequence of (ii).

To show (iv) assume that $\Xi := K_X + qA$ is of order n. By our condition $qu + nv = 1$, where $u, v \in \mathbb{Z}$. Put $A' := A - u\Xi$. Then $qA' = qA - qu\Xi = qA - \Xi \sim -K_X$. Hence, $q = \mathrm{qW}(X)$ by (i) and (iii). $\qquad\square$

Lemma 3.3. *Let X be a Fano threefold with terminal singularities.*

    (i) *$\mathrm{q\mathbb{Q}}(X) \in \{1, \dots, 11, 13, 17, 19\}$.*
    (ii) *If $\mathrm{q\mathbb{Q}}(X) \geq 5$, then $-K_X^3 \leq 125/2$.*

Proof. Denote $q := \mathrm{q\mathbb{Q}}(X)$ and write, as usual, $-K_X \sim_{\mathbb{Q}} qA$. Thus $n(K_X + qA) \sim 0$ for some positive integer n. The element $K_X + qA$ defines a cyclic étale in codimension one cover $\pi \colon X' \to X$ so that X' is a Fano threefold with terminal singularities and $K_{X'} + qA' \sim 0$, where $A' := \pi^* A$. Let $\sigma \colon X'' \to X'$ be a $\mathbb{Q}$-factorialization. (If X' is $\mathbb{Q}$-factorial, we take $X'' = X'$). Run K-MMP on X'': $\psi \colon X'' \dashrightarrow \bar{X}$. At the end we get a Mori-Fano fiber space $\bar{X} \to Z$. Let $A'' := \sigma^{-1}(A')$ and $\bar{A} := \psi_* A''$. Then $-K_{\bar{X}} \sim q\bar{A}$. If $\dim Z > 0$, then for a general fiber F of $\bar{X}/Z$, we have $-K_F \sim q\bar{A}|_F$. This is impossible because $q > 3$. Thus $\dim Z = 0$ and $\bar{X}$ is a $\mathbb{Q}$-Fano.

(i) By Lemma 3.2 the number q divides $\mathrm{qW}(\bar{X})$. On the other hand, by Theorem 1.1 we have $\mathrm{qW}(\bar{X}) \in \{1, \dots, 11, 13, 17, 19\}$. This proves (i).

To show (ii) we note that $-K_{\bar{X}}^3 \geq -K_{X''}^3 = -K_{X'}^3 \geq -K_{X''}^3$. Here the first inequality holds because for Fanos (with at worst log terminal singularities) the number $-\frac{1}{6}K^3$ is nothing but the leading term in the asymptotic Riemann-Roch and $\dim |-tK_{X''}| \leq \dim |-tK_{\bar{X}}|$. Now the assertion of (ii) follows from Theorem 1.2. $\qquad\square$

From Lemmas 3.2 and 3.3 we have

Corollary 3.4. *Let X be a Fano threefold with terminal singularities.*

(i) *If $-K_X \sim qL$ for some Weil divisor L and $q \geq 5$, then $q = \mathrm{qW}(X)$.*
(ii) *If $-K_X \sim_{\mathbb{Q}} qL$ for some Weil divisor L and $q \geq 5$, then $q = \mathrm{q}\mathbb{Q}(X)$.*

Lemma 3.5 (cf. [Suz04]). *Let X be a Fano threefold with terminal singularities and let $q := \mathrm{qW}(X)$. Assume that $\mathrm{qW}(X) \geq 8$. Then one of the following holds:*

$$q = 8, \quad \mathbf{B} = (3^2, 5), \ (3^2, 5, 9), \ (3, 5, 11), \ (3, 7), \ (3, 9), \ (5, 7),$$
$$(7, 11), \ (7, 13), \ (11),$$
$$q = 9, \quad \mathbf{B} = (2, 4, 5), \ (2^3, 5, 7), \ (2, 5, 13),$$
$$q = 10, \quad \mathbf{B} = (7, 11),$$
$$q = 11, \quad \mathbf{B} = (2, 3, 5), \ (2, 5, 7), \ (2^2, 3, 4, 7),$$
$$q = 13, \quad \mathbf{B} = (3, 4, 5), \ (2, 3^2, 5, 7),$$
$$q = 17, \quad \mathbf{B} = (2, 3, 5, 7),$$
$$q = 19, \quad \mathbf{B} = (3, 4, 5, 7).$$

In all cases the group $\mathrm{Cl}(X)$ is torsion free.

Proof. We use a computer program written in PARI [PARI]. Below is the description of our algorithm.

Step 1. By Theorem 2.7 we have $\sum_{P \in \mathbf{B}}(1 - 1/r_P) \leq 24$. Hence there is only a finite (but very huge) number of possibilities for the basket $\mathbf{B} = \{[r_P, b_P]\}$. In each case we know $-K_X \cdot c_2(X)$ from (2.6). Let $r := \mathrm{lcm}(\{r_P\})$ be the Gorenstein index of X.

Step 2. By Lemma 3.3 $\mathrm{q}\mathbb{Q}(X) \in \{8, \dots, 11, 13, 17, 19\}$. Moreover, the condition $\gcd(q, r) = 1$ (see Lemma 3.1) eliminates some possibilities.

Step 3. In each case we compute A^3 and $-K_X^3 = q^3 A^3$ by formula (2.5). Here, for $D = -A$, the number i_P is uniquely determined by $q i_P \equiv b_P \mod r_P$ and $0 \leq i_P < r_P$. Further, we check the condition $r A^3 \in \mathbb{Z}$ (Lemma 3.1) and the inequality $-K_X^3 \leq 125/2$ (Lemma 3.3).

Step 4. Finally, by the Kawamata-Viehweg vanishing theorem we have $\chi(tA) = h^0(tA)$ for $-q < t$. We compute $\chi(tA)$ by using (2.4) and check conditions $\chi(tA) = 0$ for $-q < t < 0$ and $\chi(tA) \geq 0$ for $t > 0$.

At the end we get our list. To prove the last assertion assume that $\mathrm{Cl}(X)$ contains an n-torsion element Ξ. Clearly, we also may assume that n is prime. By Proposition 2.9 we have $\sum_{n|r_i} r_i \geq 16$. Moreover, $\sum_{n|r_i} r_i \geq 18$ if $n = 3$. This does not hold in all cases of our list. $\qquad\square$

7

Proposition 3.6. *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}\mathbb{Q}(X) \geq 9$. Let $q := \mathrm{q}\mathbb{Q}(X)$ and let $-K_X \sim_{\mathbb{Q}} qA$. Then the group $\mathrm{Cl}(X)$ is torsion free, $\mathrm{qW}(X) = \mathrm{q}\mathbb{Q}(X)$, and one of the following holds:*

q	$\mathbf{B}$	A^3	$\dim|kA|$							
			$\|A\|$	$\|2A\|$	$\|3A\|$	$\|4A\|$	$\|5A\|$	$\|6A\|$	$\|7A\|$	$\|-K\|$
9	$(2,4,5)$	$\frac{1}{20}$	0	1	2	4	6	8	11	19
9	$(2,2,2,5,7)$	$\frac{1}{70}$	-1	0	0	1	1	2	3	5
10	$(7,11)$	$\frac{2}{77}$	-1	0	1	1	3	4	6	13
11	$(2,3,5)$	$\frac{1}{30}$	0	1	2	3	5	7	9	23
11	$(2,5,7)$	$\frac{1}{70}$	0	0	0	1	2	3	4	10
11	$(2,2,3,4,7)$	$\frac{1}{84}$	-1	0	0	1	1	2	3	8
13	$(3,4,5)$	$\frac{1}{60}$	0	0	1	2	3	4	5	19
13	$(2,3,3,5,7)$	$\frac{1}{210}$	-1	-1	0	0	0	1	1	5
17	$(2,3,5,7)$	$\frac{1}{210}$	-1	0	0	0	1	1	2	12
19	$(3,4,5,7)$	$\frac{1}{420}$	-1	-1	0	0	0	0	1	8

Proof. First we claim that $\mathrm{qW}(X) = \mathrm{q}\mathbb{Q}(X)$. Assume the converse. Then, as in the proof of Lemma 3.3, the class of $K_X + qA$ is a non-trivial n-torsion element in $\mathrm{Cl}(X)$ defining a global cover $\pi \colon X' \to X$. We have $K_{X'} + qA' \sim 0$, where $A' = \pi^*A$. Hence X' is such as in Lemma 3.5 and by Corollary 3.5 we have $\mathrm{Cl}(X') \simeq \mathbb{Z} \cdot A'$ and $\mathrm{qW}(X') = \mathrm{q}\mathbb{Q}(X') \geq q$. The Galois group $\boldsymbol{\mu}_n$ acts naturally on X'. Consider, for example, the case $q = 11$ and $\mathbf{B}_{X'} = (2,3,5)$ (all other cases are similar). Then X' has three cyclic quotient singularities whose indices are 2, 3, and 5. These points must be $\boldsymbol{\mu}_n$-invariant. Hence the variety X has cyclic quotient singularities of indices $2n$, $3n$, and $5n$. By Lemma 3.2 we have $\gcd(q, n) \neq 1$. In particular, $n \geq 11$. This contradicts (2.8). Therefore, $\mathrm{qW}(X) = \mathrm{q}\mathbb{Q}(X)$ and so X is such as in Lemma 3.5.

Now we have to exclude only the case $q = 9$, $\mathbf{B} = (2,5,13)$. But in this case by (2.6) and (2.5) we have $A^3 = 9/130$ and $-K_X \cdot c_2 = 621/130$. On the other hand, by Kawamata-Bogomolov's bounds [Kaw92a] we have $2673/130 = (4q^2 - 3q)A^3 \leq 4K_X \cdot c_2 = 1242/65$ [Suz04, Proposition 2.2]. The contradiction shows that this case is impossible. Finally, the values of A^3 and dimensions of $|kA|$ are computed by using (2.5) and (2.4). $\square$

Corollary 3.7. *Let X be a $\mathbb{Q}$-Fano threefold satisfying assumptions of* (i)-(v) *of Theorem* 1.4. *Then X has only cyclic quotient singularities.*

Proof. Indeed, in these cases the indices of points in the basket **B** are distinct numbers and moreover **B** contains no pairs of points of indices 2 and 4. Then the assertion follows [Mor85a], or [Rei87] □

Corollary 3.8. *Let X be a $\mathbb{Q}$-Fano threefold with* $q\mathbb{Q}(X) \geq 9$. *Then* $\dim |A| \leq 0$.

Computer computations similar to that in Lemma 3.5 allow us to prove the following.

Lemma 3.9. *Let X be a Fano threefold with terminal singularities, let $q := q\mathrm{W}(X)$, and let $A := -\frac{1}{q}K_X$.*

  (i) *If $q \geq 5$ and $\dim |A| > 1$, then $q = 5$, $\mathbf{B} = (2)$, and $A^3 = 1/2$.*
  (ii) *If $q \geq 7$ and $\dim |A| > 0$, then $q = 7$, $\mathbf{B} = (2,3)$, $A^3 = 1/6$.*

3.10. Proof of (vi) and (vii) of Theorem 1.4. (vii) Apply Lemma 3.9. Then the result is well-known: in fact, $2A$ is Cartier and by Riemann-Roch $\dim |2A| = 6 = \dim X + 3$. Hence X is a variety of Δ-genus zero [Fuj75], i.e., a variety of minimal degree. Then $X \simeq \mathbb{P}(1^3, 2)$.

  (vi) Put $q := q\mathbb{Q}(X)$, $\Xi := K_X + qA$, and $\Xi_1 := A - A_1$. By our assumption $n\Xi \sim n\Xi_1 \sim 0$ for some integer n. If either $\Xi \not\sim 0$ or $\Xi_1 \not\sim 0$, then elements Ξ and Ξ' define an étale in codimension one finite cover $\pi\colon X' \to X$ such that $K_{X'} + qA' \sim 0$ and $A' \sim A_1'$, where $A' := \pi^*A$ and $A_1' := \pi^*A_1$. If $\Xi \sim \Xi_1 \sim 0$, we put $X' = X$. In both cases, the following enequalities hold: $q\mathrm{W}(X') \geq 7$ and $\dim |A'| \geq 1$. By Lemma 3.9 we have $\mathbf{B}(X') = (2,3)$ and $q\mathbb{Q}(X') = q\mathrm{W}(X') = 7$. Note that the Gorenstein index of X' is strictly less than $q\mathrm{W}(X')$. In this case, $X' \simeq \mathbb{P}(1^2, 2, 3)$ according to [San96]. [†] Now it is sufficient to show that π is an isomorphism. Assume the converse. By our construction, there is an action of a cyclic group $\boldsymbol{\mu}_p \subset \mathrm{Gal}(X'/X)$, p is prime, such that π is decomposed as $\pi\colon X' \to X'/\boldsymbol{\mu}_p \to X$. Here $X'/\boldsymbol{\mu}_p$ is a $\mathbb{Q}$-Fano threefold and there is a torsion element of $\mathrm{Cl}(X'/\boldsymbol{\mu}_p)$ which is not Cartier exactly at points where $X' \to X'/\boldsymbol{\mu}_p$ is not étale. There are exactly four such points and two of them are points of indices 2 and 3. Thus the basket of $X'/\boldsymbol{\mu}_p$ consists of points of indices p, p, $2p$, and $3p$. This contradicts Proposition 2.9.

Lemma 3.11. *Let X be a $\mathbb{Q}$-Fano threefold with $q := q\mathbb{Q}(X)$. If there are three effective different Weil divisors A, A_1, A_2 such that $-K_X \sim_{\mathbb{Q}} qA \sim_{\mathbb{Q}} qA_1 \sim_{\mathbb{Q}} qA_2$ and $A \not\sim A_1$, then $q \leq 5$.*

Proof. Assume that $q \geq 6$. As in 3.10 consider a cover $\pi\colon X' \to X$. Thus on X' we have $A' \sim A_1' \sim A_2'$ and $-K_{X'} \sim qA'$. Moreover, $\dim |A'| = 1$ according to Lemma 3.9. In this case, the action of $\mathrm{Gal}(X'/X)$ on the

---

[†]The result also can be easily proved by using birational transformations similar to that in §4.

pencil $|A'|$ is trivial (because there are three invariant members A', A'_1, and A'_2). But then $A \sim A_1 \sim A_2$, a contradiction. □

4. Birational construction

4.1. Let X be a $\mathbb{Q}$-Fano threefold and let A be the ample Weil divisor that generates the group $\mathrm{Cl}(X)/\sim_{\mathbb{Q}}$. Thus we have $-K_X \sim_{\mathbb{Q}} qA$. Let $\mathscr{M}$ be a mobile linear system without fixed components and let $c := \mathrm{ct}(X, \mathscr{M})$ be the canonical threshold of $(X, \mathscr{M})$. So the pair $(X, c\mathscr{M})$ is canonical but not terminal. Assume that $-(K_X + c\mathscr{M})$ is ample.

Recall that the class of K_X is a generator of the local Weil divisor class group $\mathrm{Cl}(X, P)$.

Lemma 4.2. *Let $P \in X$ be a point of index $r > 1$. Assume that $\mathscr{M} \sim -mK_X$ near P, where $0 < m < r$. Then $c \leq 1/m$.*

Proof. According to [Kaw92b] there is an exceptional divisor Γ over P of discrepancy $a(\Gamma) = 1/r$. Let $\varphi\colon Y \to X$ be a resolution. Clearly, Γ is a prime divisor on Y. Write

$$K_Y = \varphi^* K_X + \frac{1}{r}\Gamma + \sum \delta_i \Gamma_i, \quad \mathscr{M}_Y = \varphi^* \mathscr{M} - \mathrm{ord}_\Gamma(\mathscr{M})\Gamma - \mathrm{ord}_{\Gamma_i}(\mathscr{M})\Gamma_i,$$

where $\mathscr{M}_Y$ is the birational transform of $\mathscr{M}$ and Γ_i are other φ-exceptional divisors. Then

$$K_Y + c\mathscr{M}_Y = \varphi^*(K_X + cM) + (1/r - c\,\mathrm{ord}_\Gamma(\mathscr{M}))\Gamma + \ldots$$

and so $1/r - c\,\mathrm{ord}_\Gamma(\mathscr{M}) \geq 0$. On the other hand, $\mathrm{ord}_\Gamma(\mathscr{M}) \equiv m/r \mod \mathbb{Z}$ (because $mK_X + \mathscr{M} \sim 0$ near P). Hence, $\mathrm{ord}_\Gamma(\mathscr{M}) \geq m/r$ and $c \leq 1/m$. □

4.3. In the construction below we follow [Ale94]. Let $f\colon \tilde{X} \to X$ be $K + c\mathscr{M}$-crepant blowup such that $\tilde{X}$ has only terminal $\mathbb{Q}$-factorial singularities:

$$(4.4) \qquad\qquad K_{\tilde{X}} + c\tilde{\mathscr{M}} = f^*(K_X + c\mathscr{M}).$$

As in [Ale94], we run $K + c\mathscr{M}$-MMP on $\tilde{X}$. We get the following diagram (Sarkisov link of type I or II)

$$(4.5)$$

$$\tilde{X} \dashrightarrow \bar{X}$$

$$f \swarrow \qquad \searrow g$$

$$X \qquad\qquad \hat{X}$$

where varieties $\tilde{X}$ and $\bar{X}$ have only $\mathbb{Q}$-factorial terminal singularities, $\rho(\tilde{X}) = \rho(\bar{X}) = 2$, f is a Mori extremal divisorial contraction, $\tilde{X} \dashrightarrow \bar{X}$ is a sequence of log flips, and g is a Mori extremal contraction (either divisorial or fiber type). Thus one of the following possibilities holds:

a) $\dim \hat{X} = 1$ and g is a $\mathbb{Q}$-del Pezzo fibration;

b) $\dim \hat{X} = 2$ and g is a $\mathbb{Q}$-conic bundle; or

c) $\dim \hat{X} = 3$, g is a divisorial contraction, and $\hat{X}$ is a $\mathbb{Q}$-Fano three-fold. In this case, denote $\hat{q} := \mathrm{q}\mathbb{Q}(\hat{X})$.

Let E be the f-exceptional divisor. For a divisor D on X, everywhere below $\tilde{D}$ and $\bar{D}$ denote strict birational transforms of D on $\tilde{X}$ and $\bar{X}$, respectively. If g is birational, we put $\hat{D} := g_* \bar{D}$.

Claim 4.6 ([Ale94]). *If the map is birational, then $\bar{E}$ is not an exceptional divisor. If g is of fiber type, then $\bar{E}$ is not composed of fibers.*

Proof. Assume the converse. If g is birational, this implies that the map $g \circ \chi \circ f^{-1} \colon X \dashrightarrow \hat{X}$ is an isomorphism in codimension 1. Since both X and $\hat{X}$ are Fano threefolds, this implies that $g \circ \chi \circ f^{-1}$ is in fact an isomorphism. On the other hand, the number of $K + c\mathcal{M}$-crepant divisors on $\hat{X}$ is less than that on X, a contradiction. If $\dim \hat{X} \leq 2$, then $\bar{E}$ is a pull-back of an ample Weil divisor on $\hat{X}$. But then $n\bar{E}$ is a movable divisor for some $n > 0$. This contradicts exceptionality of E. $\square$

If $|kA| \neq \varnothing$, let $S_k \in |kA|$ be a general member. Write

$$
\begin{aligned}
K_{\tilde{X}} &= f^* K_X + \alpha E, \\
\tilde{S}_k &= f^* S_k - \beta_k E, \\
\tilde{\mathcal{M}} &= f^* \mathcal{M} - \beta_0 E.
\end{aligned}
\tag{4.7}
$$

Then

$$
c = \alpha/\beta_0.
\tag{4.8}
$$

Remark 4.9. If $\alpha < 1$, then $a(E, |-K_X|) < 1$. On the other hand, $0 = K_X + |-K_X|$ is Cartier. Hence, $a(E, |-K_X|) \leq 0$ and so f is $f_*^{-1}|-K_X| \subset |-K_{\tilde{X}}|$. Therefore,

$$
\dim |-K_{\bar{X}}| = \dim |-K_{\tilde{X}}| \geq \dim |-K_X|.
$$

In our situation X has only cyclic quotient singularities (see Corollary 3.7). So, the following result is very important.

Theorem 4.10 ([Kaw96]). *Let $(Y \ni P)$ be a terminal cyclic quotient singularity of type $\frac{1}{r}(1, a, r-a)$, let $f \colon \tilde{Y} \to Y$ be a Mori divisorial contraction, and let E be the exceptional divisor. Then $f(E) = P$, f is the weighted blowup with weights $(1, a, r - a)$ and the discrepancy of E is $a(E) = 1/r$.*

We call this f the *Kawamata blowup* of P.

4.11. Notation. Assume that g is birational. Let $\bar{F}$ be the g-exceptional divisor and let $\tilde{F}$ and F be its proper transforms on $\tilde{X}$ and X, respectively. Let n be the maximal integer dividing the class of $\bar{F}$ in $\mathrm{Cl}(\bar{X})$. Let Θ be

11

an ample Weil divisor on $\hat{X}$ that generates $\mathrm{Cl}(\hat{X})/\sim_{\mathbb{Q}}$. Write $\hat{S}_k \sim_{\mathbb{Q}} s_k\Theta$ and $\hat{E} \sim_{\mathbb{Q}} e\Theta$, where $s_k,\, e \in \mathbb{Z}$, $s_k \geq 0$, $e \geq 1$. Note that $s_k = 0$ if and only if $\bar{S}_k$ is contracted by g.

Lemma 4.12. *In the above notation assume that the group $\mathrm{Cl}(X)$ is torsion free. Write $F \sim dA$, where $d \in \mathbb{Z}$, $d \geq 1$. Then $\mathrm{Cl}(\hat{X}) \simeq \mathbb{Z} \oplus \mathbb{Z}_n$ and $d = ne$.*

Proof. Write $\bar{F} \sim n\bar{G}$, where $\bar{G}$ is an integral Weil divisor. Then $\bar{E} \sim e\bar{\Theta} + k\bar{G}$ for some $k \in \mathbb{Z}$ and $\mathrm{Cl}(\hat{X}) \simeq \mathrm{Cl}(\bar{X})/\bar{F}\mathbb{Z} \simeq \mathbb{Z} \oplus \mathbb{Z}_n$. We have

$$\mathbb{Z}_d \simeq \mathrm{Cl}(X)/\langle F \rangle \simeq \mathrm{Cl}(\bar{X})/\langle \bar{E}, \bar{F} \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}/\langle e\bar{\Theta} + k\bar{G},\, n\bar{G}\rangle.$$

Since the last group is of order ne, we have $d = ne$. $\qquad\square$

From now until the end of this section we consider the case where $\hat{X}$ is a surface.

Lemma 4.13. *Assume that $\hat{X}$ is a surface. Then $\hat{X}$ is a del Pezzo surface with Du Val singularities of type A_n. The linear system $|-K_{\hat{X}}|$ is base point free. If moreover the group $\mathrm{Cl}(X)$ is torsion free, then so is $\mathrm{Cl}(\hat{X})$ and there are only the following possibilities:*
- (i) $K_{\hat{X}}^2 = 9$, $\hat{X} \simeq \mathbb{P}^2$;
- (ii) $K_{\hat{X}}^2 = 8$, $\hat{X} \simeq \mathbb{P}(1^2, 2)$;
- (iii) $K_{\hat{X}}^2 = 6$, $\hat{X} \simeq \mathbb{P}(1, 2, 3)$;
- (iv) $K_{\hat{X}}^2 = 5$, $\hat{X}$ *has a unique singular point, point of type A_4.*

Proof. By the main result of [MP08b] the surface $\hat{X}$ has only Du Val singularities of type A_n. Since $\rho(\hat{X}) = 1$ and $\hat{X}$ is uniruled, $-K_{\hat{X}}$ is ample. Further, since both $\bar{X}$ and $\hat{X}$ have only isolated singularities and $\mathrm{Pic}(\bar{X}/\hat{X}) \simeq \mathbb{Z}$, there is a well-defined injective map $g^*\colon \mathrm{Cl}(\hat{X}) \to \mathrm{Cl}(\bar{X})$. Hence the group $\mathrm{Cl}(\hat{X})$ is torsion free whenever so is $\mathrm{Cl}(X)$. The remaining part follows from the classification of del Pezzo surfaces with Du Val singularities (see, e.g., [MZ88]). $\qquad\square$

Proposition 4.14. *In the above notation, let $\hat{X}$ is a surface. Let $\Gamma \in |-K_{\hat{X}}|$ and let $G := g^{-1}(\Gamma)$. Suppose that there are two prime divisors D_1 and D_2 such that $\varphi(D_i) = Z$ and $K_{\hat{X}} + D_1 + D_2 + G \sim 0$. Then the pair $(\bar{X}, D_1 + D_2)$ is canonical. If furthermore the surface $\hat{X}$ is toric, then so are $\bar{X}$ and X.*

Proof. Clearly, we may replace Γ with a general member of $|-K_{\hat{X}}|$. Note that G is an elliptic ruled surface and $K_G + D_1|_G + D_2|_G \sim 0$. Hence divisors $D_1|_G$ and $D_2|_G$ are disjointed sections. This shows that $D_1 \cap D_2$ is either empty or consists of fibres. Assume that $D_1 \cap D_2 \neq \varnothing$. We can take Γ so

that $G \cap D_1 \cap D_2 = \varnothing$. By adjunction $-K_{D_1} \sim \bar{G}|_{D_1} + D_2|_{D_1}$. Since D_1 is a rational surface (birational to $\hat{X}$), $\bar{G}|_{D_1} + D_2|_{D_1}$ must be connected, a contradiction. Thus, $D_1 \cap D_2 = \varnothing$.

Therefore both divisors D_1 and D_2 contain no fibers and so $D_1 \simeq D_2 \simeq \hat{X}$. Then the pair $(\bar{X}, D_1 + D_2)$ is PLT by the Inversion of Adjunction. Since $K_{\bar{X}} + D_1 + D_2$ is Cartier, this pair must be canonical. The second assertion follows by Corollary 4.17 below. $\square$

Lemma 4.15. *Let $\varphi \colon Y \to Z$ be a $\mathbb{Q}$-conic bundle (we assume that Y is $\mathbb{Q}$-factorial and $\rho(Y/Z) = 1$). Suppose that there are two prime divisors D_1 and D_2 such that $\varphi(D_i) = Z$, the log divisor $K_Y + D_1 + D_2$ is φ-linearly trivial and canonical. Suppose furthermore that Z is singular and let $o \in Z$ be a singular point. Then $o \in Z$ is of type A_{r-1} for some $r \geq 2$ and there is a Sarkisov link*

where σ is the Kawamata blowup of a cyclic quotient singularity $\frac{1}{r}(1, a, r-a)$ over o, χ is a sequence of flips, $\bar{\varphi}$ is a $\mathbb{Q}$-conic bundle with $\rho(\bar{Y}/\bar{Z}) = 1$, and δ is a crepant contraction of an irreducible curve to o. Moreover, if $\bar{D}_i$ is the proper transform of D_i on $\bar{Y}$, then the divisor $K_{\bar{Y}} + \bar{D}_1 + \bar{D}_2$ is linearly trivial over Z and canonical.

Proof. Regard Y/Z as an algebraic germ over o. Since D_i are generically sections, the fibration φ has no discriminant curve. By [MP08c] the central fiber $C := \varphi^{-1}(o)_{\mathrm{red}}$ is irreducible and by the main result of [MP08b] Y/Z is toroidal, that is, it is locally analytically isomorphic to a toric contraction. In particular, X has exactly two singular points at $C \cap D_i$ and these points are cyclic quotients of types $\frac{1}{r}(1, a, r - a)$ and $\frac{1}{r}(-1, a, r - a)$, respectively, for some a with $\gcd(r, a) = 1$.

Now consider the Kawamata blowup of $C \cap D_1$. Let E be the exceptional divisor and let $\tilde{D}_i$ be the proper transform of D_i. Since $K_{\tilde{Y}} = \varphi^* K_Y + \frac{1}{r} E$ and the pair $(Y, D_1 + D_2)$ is canonical, we have

$$K_{\tilde{Y}} + \tilde{D}_1 + \tilde{D}_2 = \varphi^*(K_Y + D_1 + D_2).$$

It is easy to check locally that the proper transform $\tilde{C}$ of the central fiber C does not meet $\tilde{D}_1$. Moreover, $\tilde{C} \cap E$ is a smooth point of $\tilde{Y}$ and E. Thus we have $\tilde{D}_1 \cdot \tilde{C} = 0$, $E \cdot \tilde{C} = 1$, and $\tilde{D}_2 \cdot \tilde{C} = D_2 \cdot C = 1/r$. Hence, $K_{\tilde{Y}} \cdot \tilde{C} = -1/r$. Since the set-theoretical fiber over o in $\tilde{Y}$ coincides with

13

$E \cup \tilde{C}$, the divisor $-K_{\tilde{Y}}$ is ample over Z and $\tilde{C}$ generates a (flipping) extremal ray R. Run the MMP over Z in this direction, i.e., starting with R. Assume that we end up with a divisorial contraction $\bar{\varphi} \colon \bar{Y} \to \bar{Z}$. Then $\bar{\varphi}$ must contract the proper transform $\bar{E}$ of E. Here $\bar{Z}/Z$ is a Mori conic bundle and the map $Y \dashrightarrow \bar{Z}$ is an isomorphism in codimension one, so it is an isomorphism. Moreover, $\bar{Z}/Z$ has a section, the proper transforms of D_i. Hence the fibration $\bar{Z}/Z$ is toroidal over o. Consider Shokurov's difficulty [Sho85]

$$\mathrm{d}(W) := \#\{\text{exceptional divisors of discrepancy} < 1\}.$$

Then $\mathrm{d}(Y) = \mathrm{d}(\bar{Z}) = 2(r-1)$. On the other hand,

$$\mathrm{d}(\bar{Z}) - 1 \le \mathrm{d}(\bar{Y}) < \mathrm{d}(\tilde{Y}) = r - 1 + a - 1 + r - a - 1 = 2r - 3$$

(because the map $\tilde{Y} \dashrightarrow \bar{Y}$ is not an isomorphism). The contradiction shows that our MMP ends up with a $\mathbb{Q}$-conic bundle. Clearly, the divisor $K_{\bar{Y}} + \bar{D}_1 + \bar{D}_2$ is linearly trivial and canonical. By [MP08b] the surface $\bar{Z}$ has at worst Du Val singularities of type A. Hence the morphism δ is crepant [Mor85b]. $\qquad\square$

Corollary 4.16. *In the above notations assume that $\bar{Y}$ is a toric variety. Then so is Y.*

Corollary 4.17. *Notation as in Lemma* 4.15. *Assume that the base surface Z is toric. Then so is Y.*

Proof. Induction by the number e of crepant divisors of Z. If $e = 0$, then Y is smooth and $Y \simeq \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a decomposable rank-2 vector bundle on Z. $\qquad\square$

5. CASE $\mathrm{q}\mathbb{Q}(X) = 10$

Consider the case $\mathrm{q}\mathbb{Q}(X) = 10$. By Proposition 3.6 the group $\mathrm{Cl}(X)$ is torsion free and $\mathbf{B} = (7, 11)$. For $r = 7$ and 11, let P_r be a (unique) point of index r. In notation of §4, take $\mathscr{M} := |3A|$. Since $\dim |2A| = 0$, the pencil $\mathscr{M}$ has no fixed components. Apply Construction (4.5). Near P_{11} we have $A \sim -10K_X$, so $\mathscr{M} \sim -8kK_X$. By Lemma 4.2 we get $c \le 1/8$. In particular, the pair $(X, \mathscr{M})$ is not canonical. Take divisor $S_2 \in |2A|$ and a general member $S_3 \in \mathscr{M}$. For some $a_1, a_2 \in \mathbb{Z}$ we can write

$$
\begin{aligned}
K_{\tilde{X}} + 5\tilde{S}_2 &= f^*(K_X + 5S_2) - a_1 E & \sim\ -a_1 E, \\
K_{\tilde{X}} + 2\tilde{S}_2 + 2\tilde{S}_3 &= f^*(K_X + 2S_2 + 2S_3) - a_2 E & \sim\ -a_2 E.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
K_{\bar{X}} + 5\bar{S}_2 + a_1 \bar{E} &\sim\ 0, \\
K_{\bar{X}} + 2\bar{S}_2 + 2\bar{S}_3 + a_2 \bar{E} &\sim\ 0,
\end{aligned}
\tag{5.1}
$$

where $\dim|S_2| = 0$ and $\dim|S_3| = 1$. Using (4.7) we obtain

$$
\begin{aligned}
5\beta_2 &= a_1 + \alpha, \\
2\beta_2 + 2\beta_3 &= a_2 + \alpha.
\end{aligned}
$$
(5.2)

Since $S_3 \in \mathcal{M}$ is a general member, by (4.8) we have $c = \alpha/\beta_3 \le 1/8$, so $8\alpha \le \beta_3$ and $a_2 \ge 15\alpha + 2\beta_2$.

5.3. First we consider the case where $f(E)$ is either a curve or a Gorenstein point on X. Then α and β_k are non-negative integers. In particular, $a_2 \ge 15$. From (5.1) we obtain that g is birational. Indeed, otherwise restricting the second relation of (5.1) to a general fiber V we get that $-K_V$ is divisible by some number $a' \ge a_2 \ge 15$. This is impossible. Thus $\hat{X}$ is a $\mathbb{Q}$-Fano. Again from (5.1) we get $q\mathbb{Q}(\hat{X}) \ge 15$. Moreover, $\hat{E} \sim_{\mathbb{Q}} \Theta$. In particular, $|\Theta| \ne \varnothing$. This contradicts Proposition 3.6.

5.4. Therefore $f(E)$ is a non-Gorenstein point P_r of index $r = 7$ or 11. By Theorem 4.10 $\alpha = 1/r$. Near P_r we can write $A \sim -l_r K_X$, where $l_r \in \mathbb{Z}$ and $10 l_r \equiv 1 \mod r$. Then $S_k + k l_r K_X$ is Cartier near P_r. Therefore, $\beta_k \equiv k l_r \mod \mathbb{Z}$ and we can write $\beta_k = k l_r / r + m_k$, where $m_k = m_{k,r} \in \mathbb{Z}_{\ge 0}$. Explicitly, we have the following values of α, β_k, and a_k:

r	α	β_2	β_3	a_1	a_2
7	$\frac{1}{7}$	$\frac{3}{7} + m_2$	$\frac{1}{7} + m_3$	$2 + 5m_2$	$1 + 2m_2 + 2m_3$
11	$\frac{1}{11}$	$\frac{9}{11} + m_2$	$\frac{8}{11} + m_3$	$4 + 5m_2$	$3 + 2m_2 + 2m_3$

Claim 5.5. *If $r = 7$, then $m_3 \ge 1$.*

Proof. Follows from $c = \alpha/\beta_3 \le 1/8$. $\square$

If g is not birational, then $a_i \le 3$, so $r = 7$. By the above claim we have $a_2 \ge 3$. In this case, g is a generically $\mathbb{P}^2$-bundle and $m_2 = 0$ (because $-K_{\bar{X}}$ restricted to a general fiber is divisible by $a_2 \ge 3$). On the other hand, $a_1 = 2$ and $\bar{S}_2$ is g-vertical, a contradiction. Thus g is birational. Since $\bar{S}_3$ is moveable, $s_3 \ge 1$. Put

$$
u := s_2 + e m_2, \qquad v := s_3 + e m_3.
$$

5.6. Case: $r = 11$. Then

$$
\begin{aligned}
\hat{q} &= 5 s_2 + (4 + 5 m_2) e &&= 5u + 4e, \\
\hat{q} &= 2 s_2 + 2 s_3 + (3 + 2 m_2 + 2 m_3) e &&= 2u + 2v + 3e.
\end{aligned}
$$
(5.7)

Assume that $u = 0$. Then $\hat{q} = 4e$. The only solution of (5.7) is the following: $\hat{q} = 8$, $v = 1$, $e = 2$. Hence, $s_2 = 0$ and $s_3 = 1$. In particular, $\dim|\Theta| \ge \dim|S_3| = 1$. On the other hand, by Lemma 4.12 the group $\mathrm{Cl}(\hat{X})$ is torsion free and by Lemma 3.9 the divisor Θ is not moveable, a contradiction.

15

Therefore, $u \geq 1$. By the first relation in (5.7) $\hat{q} \geq 9$. Hence the group $\mathrm{Cl}(\hat{X})$ is torsion free. Then by Lemma 4.12 we have $F \sim eA$. Since $|A| = \varnothing$, $e \geq 2$. Again by (5.7) $\hat{q} \geq 13$ and e is odd. Thus, $e = 3$, $u = 1$, and $\hat{q} = 17$. Further, $s_3 + em_3 = v = 3$ and $s_3 = 3$ (because $\bar{S}_3$ is moveable). By Proposition 3.6 we have $1 = \dim |S_3| \leq \dim |3\Theta| = 0$, a contradiction.

5.8. Case: $r = 7$. Recall that $m_3 \geq 1$ by Claim 5.5. Write

$$
\begin{array}{rclcl}
\hat{q} & = & 5s_2 + (2 + 5m_2)e & = & 5u + 2e, \\
\hat{q} & = & 2s_2 + 2s_3 + (1 + 2m_2 + 2m_3)e & = & 2u + 2v + e.
\end{array}
$$
(5.9)

Hence, $v = s_3 + em_3 \geq 1 + e$.

If $u = 0$, then $\hat{q} = 2e = 2v + e$, $e = 2v$, and $\hat{q} = 4v \geq 4(1+e) = 4(1+2v)$, a contradiction. If $u = 2$, then $\hat{q}$ is even ≥ 12. Again we have a contradiction.

Assume that $u \geq 3$. Using the first relation in (5.9) and Proposition 3.6 we get successively $u = 3$, $\hat{q} \geq 17$, $|\Theta| = \varnothing$, $e \geq 2$, $\hat{q} \geq 19$, $|2\Theta| = \varnothing$, $e \geq 3$, and so $\hat{q} \geq 21$, a contradiction.

Therefore, $u = 1$. Then $\hat{q} = 5 + 2e = 2 + 2v + e$ and $2v = 3 + e = 2v \geq 2 + 2e$. So, $e = 1$, $v = 2$, $\hat{q} = 7$. Since $m_3 \geq 1$, $s_3 = v - em_3 = 1$. Hence, $\hat{S}_3 \sim_\mathbb{Q} \Theta$. Since $\dim |\hat{S}_3| \geq 1$, by (vi) of Theorem 1.4 we have $\hat{X} \simeq \mathbb{P}(1^2, 2, 3)$. In particular, the group $\mathrm{Cl}(\hat{X})$ is torsion free. By Lemma 4.12 the divisor F generates the group $\mathrm{Cl}(X)$. This contradicts $|A| = \varnothing$.

The last contradiction finishes the proof of (v) of Theorem 1.4.

6. Case $q\mathbb{Q}(X) = 11$ and $\dim |-K_X| \geq 11$

In this section we consider the case $q\mathbb{Q}(X) = 11$ and $\dim |-K_X| \geq 11$. By Proposition 3.6 the group $\mathrm{Cl}(X)$ is torsion free and $\mathbf{B} = (2, 3, 5)$. For $r = 2$, 3, 5, let P_r be a (unique) point of index r. In notation of §4, take $\mathscr{M} := |2A|$. Since $0 = \dim |A| > \dim \mathscr{M} = 1$, the linear system $\mathscr{M}$ has no fixed components. Apply Construction (4.5). Near P_5 we have $A \sim -K_X$ and $\mathscr{M} \sim -2K_X$. By Lemma 4.2 we get $c \leq 1/2$. In particular, the pair $(X, \mathscr{M})$ is not canonical. It can be easily seen from Proposition 3.6 that there are reduced irreducible members $S_k \in |kA|$ for $k = 1$, 2, 3, 5.

Proposition 6.1. *In the above notation, f is the Kawamata blwup of P_5 and $\hat{X}$ is a del Pezzo surface with Du Val singularities with $K_{\hat{X}}^2 = 5$ or 6. Moreover, for $k = 1$, 2 and 3, the image $C_k := g(\bar{S}_k)$ is a curve on $\hat{X}$ with $-K_{\hat{X}} \cdot C_k = k$.*

Proof. Similar to (5.1)-(5.2) we have for some $a_1, a_2, a_3 \in \mathbb{Z}$:

$$
\begin{array}{rcl}
K_{\bar{X}} + 11\bar{S}_1 + a_1\bar{E} & \sim & 0, \\
K_{\bar{X}} + \bar{S}_1 + 5\bar{S}_2 + a_2\bar{E} & \sim & 0, \\
K_{\bar{X}} + 2\bar{S}_1 + 3\bar{S}_3 + a_3\bar{E} & \sim & 0,
\end{array}
$$
(6.2)

$$\text{(6.3)} \qquad \begin{aligned} 11\beta_1 &= a_1 + \alpha, \\ \beta_1 + 5\beta_2 &= a_2 + \alpha, \\ 2\beta_1 + 3\beta_3 &= a_3 + \alpha. \end{aligned}$$

Since $S_2 \in \mathscr{M}$ is a general member, by (4.8) we have $c = \alpha/\beta_2 \le 1/2$, so $2\alpha \le \beta_2$ and $a_2 \ge 9\alpha + \beta_1$. Since $2S_1 \sim S_2$, we have $2\beta_1 \ge \beta_2$. Thus $\beta_1 \ge \alpha$ and $a_1, a_2 \ge 10\alpha$.

First we consider the case where $f(E)$ is either a curve or a Gorenstein point on X. Then α and β_k are integers, so $a_1, a_2 \ge 10$. From (6.2) we obtain that g is birational. Moreover, $\hat{q} \ge 15$, the group $\mathrm{Cl}(\hat{X})$ is torsion free, and $\hat{E} \sim \Theta$. In particular, $|\Theta| \ne \varnothing$. This contradicts Proposition 3.6.

6.4. Therefore $P := f(E)$ is a non-Gorenstein point of index $r = 2, 3$ or 5. As in 5.4 we have the following values of β_k and a_k:

r	β_1	β_2	β_3	a_1	a_2	a_3
2	$\frac{1}{2} + m_1$	m_2	$\frac{1}{2} + m_3$	$5 + 11m_1$	$m_1 + 5m_2$	$2 + 2m_1 + 3m_3$
3	$\frac{2}{3} + m_1$	$\frac{1}{3} + m_2$	m_3	$7 + 11m_1$	$1 + m_1 + 5m_2$	$1 + 2m_1 + 3m_3$
5	$\frac{1}{5} + m_1$	$\frac{2}{5} + m_2$	$\frac{3}{5} + m_3$	$2 + 11m_1$	$2 + m_1 + 5m_2$	$2 + 2m_1 + 3m_3$

Claim 6.5. *If $r = 2$ or 3, then $m_2 \ge 1$.*

Proof. Follows from $1/2 \ge c = \alpha/\beta_2 = 1/r\beta_2$. $\qquad\square$

Assume that g is birational. By Proposition 3.6 and Remark 4.9 we have $\dim|-K_{\hat{X}}| \ge |-K_X| = 23$. So, $\hat{q} \le 11$. If $\bar{S}_1$ is not contracted, then by the first relation in (6.2) we have $\hat{q} \ge 11 + a_1 \ge 13$, a contradiction. Therefore the divisor $\bar{S}_1$ is contracted. By Lemma 4.12 the group $\mathrm{Cl}(\hat{X})$ is torsion free and $\hat{E} \sim \Theta$. Hence, $\hat{q} = a_1 \le 7$, $m_1 = 0$, and $r \ne 5$. But then $m_2 \ge 1$ (see Claim 6.5) and $a_2 \ge 5$. This contradicts the second relation in (6.2).

Therefore g is of fiber type. Restricting (6.2) to a general fiber we get $a_i \le 3$. Thus, $r = 5$ and $a_1 = a_2 = a_3 = 2$. Moreover, divisors $\bar{S}_1$, $\bar{S}_2$, and $\bar{S}_3$ are g-vertical. Since $\bar{S}_3$ is irreducible and $\dim|\bar{S}_3| = 2$, $\hat{X}$ cannot be a curve. Therefore $\hat{X}$ is a surface and the images $g(\bar{S}_1)$, $g(\bar{S}_2)$, and $g(\bar{S}_3)$ are curves. Since $\dim|\bar{S}_1| = 0$, we have $\dim|g(\bar{S}_1)| = 0$. Hence, $K_{\hat{X}}^2 \le 6$ and $g(\bar{S}_1)$ is a line on $\hat{X}$. By Lemma 4.13 there are only two possibilities: $\hat{X} \simeq \mathbb{P}(1, 2, 3)$ and $\hat{X}$ is an A_4-del Pezzo surface. $\qquad\square$

6.6. Consider the case where $\hat{X}$ is an A_4-del Pezzo surface. Assume that $\bar{S}_6$ is g-vertical. By Riemann-Roch for Weil divisors on surfaces with Du Val singularities [Rei87] we have $\dim|\bar{S}_6| = \dim|g(\bar{S}_6)| = 6$. On the other

hand, $\dim |\bar{S}_6| = \dim |S_6| = 7$, a contradiction. Thus $g(\bar{S}_5) = \hat{X}$. Since $K_X + S_5 + S_6 \sim 0$,
$$K_{\bar{X}} + \bar{S}_5 + \bar{S}_6 + \bar{E} \sim 0.$$
Therefore $\bar{S}_6$ and $\bar{E}$ are sections of g. By Proposition 4.14 the pair $(\bar{X}, \bar{S}_6 + \bar{E})$ is canonical. Now since $\bar{S}_5$ is nef, the map $\bar{X} \dashrightarrow \tilde{X}$ is a composition of steps of the $K_{\bar{X}} + \bar{S}_6 + \bar{E}$-MMP. Hence the pair $(\tilde{X}, \tilde{S}_6 + E)$ is also canonical. In particular, $\tilde{S}_6 \cap E = \varnothing$ and so $P_5 = f(E) \notin S_6$, a contradiction.

6.7. Now consider the case $\hat{X} \simeq \mathbb{P}(1, 2, 3)$. As above, if $g(\bar{S}_5)$ is a curve, then $\dim |g(\bar{S}_5)| = 5$ and $g(\bar{S}_5) \sim 5g(\bar{S}_1)$. On the other hand, $g(\bar{S}_5) \sim -\frac{5}{6} K_{\hat{X}}$. But then $\dim |g(\bar{S}_5)| = 4$, a contradiction. Therefore, $g(\bar{S}_5) = \hat{X}$. Similar to (6.2) we have $K_{\bar{X}} + 2\bar{S}_5 + \bar{S}_1 + a_4 \bar{E} \sim 0$. This shows that $a_4 = 0$ and $\bar{S}_5$ is a section of g. Thus we can write $K_{\bar{X}} + \bar{S}_5 + G + \bar{E} \sim 0$, where G is a g-trivial Weil divisor, i.e., $G = g^*\Gamma$ for some Weil divisor Γ. Pushing down this equality to X we get $G \sim 6\bar{S}_1$, i.e., $\Gamma \in |-K_{\hat{X}}|$. By Proposition 4.14 varieties $\bar{X}$ and X are toric. This proves (iv) of Theorem 1.4.

7. Case $\mathrm{q}\mathbb{Q}(X) = 13$ and $\dim |-K_X| \geq 6$

In this section we consider the case $\mathrm{q}\mathbb{Q}(X) = 13$ and $\dim |-K_X| \geq 6$. By Proposition 3.6 $\mathbf{B} = (3, 4, 5)$. For $r = 3$, 4, 5, let P_r be a (unique) point of index r. In notation of §4, take $\mathscr{M} := |4A|$. Since $1 = \dim |3A| > \dim \mathscr{M} = 2$, the linear system $\mathscr{M}$ has no fixed components. Apply Construction (4.5). Near P_5 we have $A \sim -2K_X$ and $\mathscr{M} \sim -3K_X$. By Lemma 4.2 we get $c \leq 1/3$. In particular, the pair $(X, \mathscr{M})$ is not canonical.

Proposition 7.1. *In the above notation, f is the Kawamata blowup of P_5, g is birational, it contracts $\bar{S}_1$, and $\hat{X} \simeq \mathbb{P}(1^3, 2)$. Moreover, $\hat{S}_3 \sim \hat{S}_4 \sim \hat{E} \sim \Theta$ and $\hat{S}_5 \sim 2\Theta$.*

Proof. Similar to (5.1)-(5.2) we have for some $a_1, a_2, a_3 \in \mathbb{Z}$:

$$
\begin{aligned}
K_{\bar{X}} + 13\bar{S}_1 + a_1 \bar{E} &\sim 0, \\
(7.2) \qquad K_{\bar{X}} + \bar{S}_1 + 4\bar{S}_3 + a_2 \bar{E} &\sim 0, \\
K_{\bar{X}} + \bar{S}_1 + 3\bar{S}_4 + a_3 \bar{E} &\sim 0,
\end{aligned}
$$

$$
\begin{aligned}
13\beta_1 &= a_1 + \alpha, \\
(7.3) \qquad \beta_1 + 4\beta_3 &= a_2 + \alpha, \\
\beta_1 + 3\beta_4 &= a_3 + \alpha.
\end{aligned}
$$

Since $S_4 \in \mathscr{M}$ is a general member, by (4.8) we have $c = \alpha/\beta_4 \leq 1/3$, $3\alpha \leq \beta_4$ and $a_3 \geq 8\alpha + \beta_1$. Since $4S_1 \sim S_4$, we have $4\beta_1 \geq \beta_4$. Thus $\beta_1 \geq \alpha$ and $a_1 \geq 12\alpha$.

First we consider the case where $f(E)$ is either a curve or a Gorenstein point on X. Then α and β_k are integers. In particular, $a_1 \geq 12$. From

18

the first relation in (7.2) we obtain that g is birational. Moreover, $\hat{q} \geq 13$ and $\hat{E} \sim \Theta$. In particular, $|\Theta| \neq \varnothing$. By Proposition 3.6 we have $\hat{q} = 13$, $a_1 = 13$, $\bar{S}_1$ is contracted, and $\alpha = 1$. This contradicts (7.3).

Therefore $P := f(E)$ is a non-Gorenstein point of index $r = 3, 4$ or 5. By Theorem 4.10 $\alpha = 1/r$. Similar to 5.4 we have (here $m_k \in \mathbb{Z}_{\geq 0}$)

r	β_1	β_3	β_4	β_5	a_1	a_2	a_3
3	$\frac{1}{3} + m_1$	m_3	$\frac{1}{3} + m_4$	$\frac{2}{3} + m_5$	$4 + 13m_1$	$m_1 + 4m_3$	$1 + m_1 + 3m_4$
4	$\frac{1}{4} + m_1$	$\frac{3}{4} + m_3$	m_4	$\frac{1}{4} + m_5$	$3 + 13m_1$	$3 + m_1 + 4m_3$	$m_1 + 3m_4$
5	$\frac{2}{5} + m_1$	$\frac{1}{5} + m_3$	$\frac{3}{5} + m_4$	m_5	$5 + 13m_1$	$1 + m_1 + 4m_3$	$2 + m_1 + 3m_4$

Claim 7.4. *If $r = 3$ or 4, then $m_4 \geq 1$.*

Proof. Follows from $1/3 \geq c = \alpha/\beta_4 = 1/r\beta_4$. $\qquad\square$

If g is not birational, then $a_1 = 3$, $r = 4$, $m_4 \geq 1$, and $a_3 \geq 3$. In this case, $a_2 = a_3 = 3$, g is a generically $\mathbb{P}^2$-bundle, and divisors $\bar{S}_1$, $\bar{S}_3$, $\bar{S}_4$ are g-vertical. Since $\dim |\bar{S}_4| > 1$ and the divisor $\bar{S}_4$ is irreducible, we have a contradiction. Therefore g is birational.

By Proposition 3.6 we have $\dim |-K_{\hat{X}}| \geq |-K_X| = 19$ and $\hat{q} \leq 13$. From the first relation in (7.2) we see that $\bar{S}_1$ is contracted. By Lemma 4.12 the group $\mathrm{Cl}(\hat{X})$ is torsion free and $\hat{E} \sim \Theta$. Moreover, $m_1 = 0$ (because $13m_1 < a_1e = \hat{q} \leq 13$). Thus $\hat{q} = a_1 = 4, 3$, and 5 in cases $r = 3, 4$, and 5, respectively.

In cases $r = 3$ and 4 we have $\hat{q} \geq 3 + a_3 \geq 6$, a contradiction. Therefore, $r = 5$, $\hat{q} = 5$, and $s_3 = s_4 = 1$. Since $\dim |\Theta| \geq 1$, by (vi) of Theorem 1.4 we have $\hat{X} \simeq \mathbb{P}(1^3, 2)$. Since $\dim |S_5| = 3$ and $\dim |\Theta| = 2$, $s_5 \geq 2$. Similar to (7.2)-(7.3) we have $K_{\bar{X}} + \bar{S}_3 + 2\bar{S}_5 + a_4\bar{E} \sim 0$, $2s_5 + a_4 = 4$, and $a_4 = \beta_3 + 2\beta_5 - \alpha = m_3 + 2m_5$. Thus, $s_5 = 2$ and $a_4 = \beta_5 = 0$, i.e., $P_5 \notin S_5$. $\qquad\square$

Lemma 7.5.   (i) $S_1 \cap S_3$ *is a reduced irreducible curve.*
      (ii) $S_1 \cap S_3 \cap S_4 = \{P_5\}$.

Proof. (i) Recall that $A^3 = 1/60$ by Proposition 3.6. Write $S_1 \cap S_3 = C + \Gamma$, where C is a reduced irreducible curve passing through P_5 and Γ is an effective 1-cycle. Suppose, $\Gamma \neq 0$. Then $1/4 = S_1 \cdot S_3 \cdot S_5 > S_5 \cdot C$. Since $P_5 \notin S_5$, $C \not\subset S_5$ and $S_5 \cdot C \geq 1/4$, a contradiction. Hence, $S_1 \cap S_3 = C$.

(ii) Assume that $S_1 \cap S_3 \cap S_4 \ni P \neq P_5$. Since $1/5 = S_1 \cdot S_3 \cdot S_4 = S_4 \cdot C$ and $P, P_5 \in S_4 \cap C$, we have $C \subset S_4$. If there is a component $C' \neq C$ of $S_1 \cap S_4$ not contained in S_5, then, as above, $1/3 = S_1 \cdot S_4 \cdot S_5 \geq S_5 \cdot C + S_5 \cdot C' \geq 1/2$, a contradiction. Thus we can write $S_1 \cap S_4 = C + \Gamma$, where Γ is an effective 1-cycle with $\mathrm{Supp}\,\Gamma \subset S_5$. In particular, $P_5 \notin \Gamma$. The divisor $12A$ is Cartier

at P_3 and P_4. We get

$$\frac{1}{5} = 12A^3 = 12A \cdot S_1 \cdot (S_4 - S_3) = 12A \cdot \Gamma \in \mathbb{Z},$$

a contradiction. $\qquad\square$

Lemma 7.6. *Let X be a $\mathbb{Q}$-Fano threefold and $D = D_1 + \cdots + D_4$ be a divisor on X, where D_i are irreducible components. Let $P \in X$ be a cyclic quotient singularity of index r. Assume that $K_X + D \sim_{\mathbb{Q}} 0$, $P \notin D_4$, $D_1 \cap D_2 \cap D_3 = \{P\}$, and $D_1 \cdot D_2 \cdot D_3 = 1/r$. Then the pair (X, D) is LC.*

Proof. Let $\pi\colon (X^\sharp, P^\sharp) \to (X, P)$ be the index-one cover. For $k = 1, 2, 3$, let $D_k^\sharp$ be the preimage of D_k and let $D^\sharp := D_1^\sharp + D_2^\sharp + D_3^\sharp$. By our assumptions $D_1^\sharp \cap D_2^\sharp \cap D_3^\sharp = \{P^\sharp\}$. Since $D_1 \cdot D_2 \cdot D_3 = 1/r$, locally near $P^\sharp$ we have $D_1^\sharp \cdot D_2^\sharp \cdot D_3^\sharp = 1$. Hence $D^\sharp$ is a simple normal crossing divisor (near $P^\sharp$). In particular, $(X^\sharp, D^\sharp)$ is LC near $P^\sharp$ and so is (X, D) near P.

Thus the pair (X, D) is LC in some neighborhood $U \ni P$. Since $D_1 \cap D_2 \cap D_3 = \{P\}$, P is a center of LC singularities for (X, D). Let H be a general hyperplane section through P. Write $\lambda D_4 \sim_{\mathbb{Q}} H$, where $\lambda > 0$. If (X, D) is not LC in $X \setminus U$, then the locus of log canonical singularities of the pair $(X, D + \epsilon H - (\lambda\epsilon + \delta)D_4)$ is not connected for $0 < \delta \ll \epsilon \ll 1$. This contradicts Connectedness Lemma [Sho92], [Kol92]. Therefore the pair (X, D) is LC. $\qquad\square$

7.7. Proof of (iii) of Theorem 1.4. By Lemma 7.6 the pair $(X, S_1 + S_3 + S_4 + S_5)$ is LC. Since $K_X + S_1 + S_3 + S_4 + S_5 \sim 0$, it is easy to see that $a(E, S_1 + S_3 + S_4 + S_5) = -1$. Thus $K_{\tilde{X}} + \tilde{S}_1 + \tilde{S}_3 + \tilde{S}_4 + \tilde{S}_5 = f^*(K_X + S_1 + S_3 + S_4 + S_5) \sim 0$. Therefore the pairs $(\bar{X}, \bar{S}_1 + \bar{S}_3 + \bar{S}_4 + \bar{S}_5 + \bar{E})$ and $(\hat{X}, \hat{S}_3 + \hat{S}_4 + \hat{S}_5 + \hat{E})$ are also LC. It follows from Proposition 7.1 and its proof that $\hat{X} \simeq \mathbb{P}(1^3, 2)$, $\hat{E} \sim \hat{S}_3 \sim \hat{S}_4 \sim \Theta$, and $\hat{S}_5 \sim 2\Theta$. We claim that $\hat{S}_3 + \hat{S}_4 + \hat{S}_5 + \hat{E}$ is a toric boundary (for a suitable choice of coordinates in $\mathbb{P}(1^3, 2)$). Let $(x_1 : x_1' : x_1'' : x_2)$ be homogeneous coordinates in $\mathbb{P}(1^3, 2)$. Clearly, we may assume that $\hat{E} = \{x_1 = 0\}$, $\hat{S}_3 = \{x_1' = 0\}$, and $\hat{S}_4 = \{\alpha x_1 + \alpha' x_1' + \alpha'' x_1'' = 0\}$ for some constants α, α', α''. Since $(\hat{X}, \hat{S}_3 + \hat{S}_4 + \hat{E})$ is LC, $\alpha'' \neq 0$ and after a coordinate change we may assume that $\hat{S}_4 = \{x_1'' = 0\}$. Further, the surface $\hat{S}_5$ is given by the equation $\beta x_2 + \psi(x_1, x_1', x_1'') = 0$, where β is a constant and ψ is a quadratic form. If $\beta = 0$, then $\hat{S}_3 \cap \hat{S}_4 \cap \hat{E} \cap \hat{S}_5 \neq \varnothing$ and the pair $(\hat{X}, \hat{S}_3 + \hat{S}_4 + \hat{S}_5 + \hat{E})$ cannot be LC. Thus $\beta \neq 0$ and after a coordinate change we may assume that $\hat{S}_5 = \{x_2 = 0\}$. Therefore $\hat{S}_3 + \hat{S}_4 + \hat{S}_5 + \hat{E}$ is a toric boundary. Then by Lemma 7.8 below the varieties $\bar{X}$, $\tilde{X}$, and X are toric. This proves (iii) of Theorem 1.4.

Lemma 7.8 (see, e.g., [McK01, 3.4]). *Let V be a toric variety and let Δ be the toric (reduced) boundary. Then every valuation ν with discrepancy -1*

with respect to $K_V + \Delta$ is toric, that is, there is a birational toric morphism $\tilde{V} \to V$ such that ν corresponds to an exceptional divisor.

8. CASE $q\mathbb{Q}(X) = 17$

Consider the case $q\mathbb{Q}(X) = 17$. By Proposition 3.6 $\mathbf{B} = (2, 3, 5, 7)$. For $r = 2, 3, 5, 7$, let P_r be a (unique) point of index r. In notation of §4, take $\mathscr{M} := |5A|$ and apply Construction (4.5). Near P_7 we have $A \sim -5K_X$ and $\mathscr{M} \sim -4K_X$. By Lemma 4.2 we get $c \leq 1/4$. In particular, the pair $(X, \mathscr{M})$ is not canonical.

Proposition 8.1. *In the above notation, f is the Kawamata blowup of P_7, g is birational, it contracts $\bar{S}_2$, and $\hat{X} \simeq \mathbb{P}(1^2, 2, 3)$. Moreover, $\hat{S}_3 \sim \hat{S}_5 \sim \Theta$, $\hat{E} \sim 2\Theta$, and $\hat{S}_7 \sim 3\Theta$.*

Proof. Similar to (5.1)-(5.2) we have for some $a_1, a_2, a_3 \in \mathbb{Z}$:

$$(8.2) \quad \begin{aligned} K_{\bar{X}} + 7\bar{S}_2 + \bar{S}_3 + a_1\bar{E} &\sim 0, \\ K_{\bar{X}} + \bar{S}_2 + 5\bar{S}_3 + a_2\bar{E} &\sim 0, \\ K_{\bar{X}} + \bar{S}_2 + 3\bar{S}_5 + a_3\bar{E} &\sim 0, \end{aligned}$$

$$(8.3) \quad \begin{aligned} 7\beta_2 + \beta_3 &= a_1 + \alpha, \\ \beta_2 + 5\beta_3 &= a_2 + \alpha, \\ \beta_2 + 3\beta_5 &= a_3 + \alpha. \end{aligned}$$

Since $S_5 \in \mathscr{M}$ is a general member, by (4.8) we have $c = \alpha/\beta_5 \leq 1/4$, so $4\alpha \leq \beta_5$ and $a_3 \geq 11\alpha + \beta_2$. Since $S_2 + S_3 \sim S_5$, we have $\beta_2 + \beta_3 \geq \beta_5 \geq 4\alpha$. Hence, $a_1 \geq 6\beta_2 + 3\alpha$ and $a_2 \geq 4\beta_3 + 3\alpha$.

First we consider the case where $f(E)$ is either a curve or a Gorenstein point on X. Then α and β_k are integers. In particular, $a_3 \geq 11$ and by the third relation in (8.2) we obtain that g is birational. Moreover, $\hat{q} \geq 11$. In particular, the group $\mathrm{Cl}(\hat{X})$ is torsion free and so $\hat{E} \geq 2\Theta$. Hence, $\hat{q} \geq 2a_3 \geq 22$, a contradiction.

Therefore $P := f(E)$ is a non-Gorenstein point of index $r = 2, 3, 5$ or 7. Similar to 5.4 we have $\alpha = 1/r$ and

r	β_2	β_3	β_5	β_7	a_1	a_2	a_3
2	m_2	$\frac{1}{2} + m_3$	$\frac{1}{2} + m_5$	$\frac{1}{2} + m_7$	$7m_2 + m_3$	$2 + m_2 + 5m_3$	$1 + m_2 + 3m_5$
3	$\frac{1}{3} + m_2$	m_3	$\frac{1}{3} + m_5$	$\frac{2}{3} + m_7$	$2 + 7m_2 + m_3$	$m_2 + 5m_3$	$1 + m_2 + 3m_5$
5	$\frac{1}{5} + m_2$	$\frac{4}{5} + m_3$	m_5	$\frac{1}{5} + m_7$	$2 + 7m_2 + m_3$	$4 + m_2 + 5m_3$	$m_2 + 3m_5$
7	$\frac{3}{7} + m_2$	$\frac{1}{7} + m_3$	$\frac{4}{7} + m_5$	m_7	$3 + 7m_2 + m_3$	$1 + m_2 + 5m_3$	$2 + m_2 + 3m_5$

Claim 8.4. (i) *If $r = 2$, then $m_5 \geq 2$ and $m_2 + m_3 \geq 2$.*
(ii) *If $r = 3$, then $m_5 \geq 1$ and $m_2 + m_3 \geq 1$.*

(iii) *If $r = 5$, then $m_5 \geq 1$.*

Proof. Note that $1/4 \geq c = \alpha/\beta_5 = 1/r\beta_5$ and $r\beta_5 \geq 4$. This gives us inequalities for m_5. The inequalities for $m_2 + m_3$ follows from $\beta_2 + \beta_3 \geq \beta_5$. $\square$

From this we have $\min(a_1, a_2, a_3) \geq 3$. Moreover, the equality $\min(a_1, a_2, a_3) = 3$ holds only if $r = 7$. Therefore the contraction g can be of fiber type only if $a_1 = 3$, $r = 7$, $m_2 = m_3 = 0$, $\min(a_1, a_2, a_3) = 3$, $r = 7$, $m_2 = m_3 = m_5 = 0$, $a_3 = 2$, and $a_2 = 1$. Then g is a del Pezzo fibration of degree 9 and by the first relation in (8.2) divisors $\hat{S}_2$ and $\hat{S}_3$ are g-vertical. But then $a_2 = 3$, a contradiction. From now on we assume that g is birational.

Since $\bar{S}_5$ is moveable, it is not contracted. Therefore, $s_5 \geq 1$. By (8.2) we have

$$\hat{q} = 7s_2 + s_3 + a_1 e,$$
$$\hat{q} = s_2 + 5s_3 + a_2 e,$$
$$\hat{q} = s_2 + 3s_5 + a_3 e.$$

Put

$$u := s_2 + em_2, \qquad v := s_3 + em_3, \qquad w := s_5 + em_5.$$

8.5. Case: $r = 2$. Then $a_3 \geq 7$ and $\hat{q} \geq 3s_5 + a_3 \geq 10$. Hence the group $\mathrm{Cl}(\hat{X})$ is torsion free. So, $e \geq 2$ and $\hat{q} \geq 3s_5 + 2a_3 \geq 17$. In this case $|\Theta| = \varnothing$. Therefore, $s_5 \geq 2$ and $\hat{q} \geq 3s_5 + 2a_3 \geq 20$, a contradiction.

8.6. Case: $r = 3$. Then

$$\hat{q} = 7s_2 + s_3 + (2 + 7m_2 + m_3)e = 7u + v + 2e,$$
$$\hat{q} = s_2 + 5s_3 + (m_2 + 5m_3)e = u + 5v,$$
$$\hat{q} = s_2 + 3s_5 + (1 + m_2 + 3m_5)e = u + 3w + e.$$

Assume that $u > 0$. Then $\hat{q} \geq 9$. Hence the group $\mathrm{Cl}(\hat{X})$ is torsion free and $e \geq 2$. Since $\dim|S_5| = 1$ and $\dim|\Theta| \leq 0$, we have $s_5 \geq 2$. Since $m_5 \geq 1$ (see Claim 8.4), we have $w \geq 4$ and $\hat{q} > 13$. In this case, $s_5 \geq 5$, a contradiction.

Therefore, $u = 0$, $m_2 = 0$, $s_3 \neq 0$, $m_3 \geq 1$, and $v \geq 2$. So, $\hat{q} = 5v \geq 10$. Then we get a contradiction by (v) of Theorem 1.4.

8.7. Case: $r = 5$. Then

$$\hat{q} = 7s_2 + s_3 + (2 + 7m_2 + m_3)e = 7u + v + 2e,$$
$$\hat{q} = s_2 + 5s_3 + (4 + m_2 + 5m_3)e = u + 5v + 4e,$$
$$\hat{q} = s_2 + 3s_5 + (m_2 + 3m_5)e = u + 3w.$$

From the first two relations we have $3u = 2v + e$ and $1 \leq u \leq 2$. Further, $\hat{q} - 4u = 3(v + e)$, so $\hat{q} \equiv u \mod 3$.

If $u = 2$, then e is even and $\hat{q} = 14 + v + 2e \geq 18$. So, $\hat{q} = 19$, a contradiction.

Thus $u = 1$, $3 = 2v + e$, and $\hat{q} = 7 + v + 2e \geq 9$. By (v) of Theorem 1.4 $\hat{q}$ is odd. Hence, v is even, $e = 3$, $v = 0$, $\hat{q} = 13$. In this case, $s_5 + 3m_5 = w = 4$. By Claim 8.4 $m_5 = s_5 = 1$. Note that the group $\mathrm{Cl}(\hat{X})$ is torsion free and $s_2 = 1$. Thus $\dim |\Theta| > 0$. This contradicts Proposition 3.6.

8.8. Case: $r = 7$. Then

$$
\begin{aligned}
\hat{q} &= 7s_2 + s_3 + (3 + 7m_2 + m_3)e &= 7u + v + 3e, \\
\hat{q} &= s_2 + 5s_3 + (1 + m_2 + 5m_3)e &= u + 5v + e, \\
\hat{q} &= s_2 + 3s_5 + (2 + m_2 + 3m_5)e &= u + 3w + 2e.
\end{aligned}
$$

Assume that $u > 0$. Then $\hat{q} \geq 10$, the group $\mathrm{Cl}(\hat{X})$ is torsion free and so $e \geq 2$, $\hat{q} \geq 13$, $u = 1$. From the first two relations we get $\hat{q} + 2 = 7v$. Hence, $v = 3$, $\hat{q} = 19$, $e = 3$, and $s_2 = 0$. This contradicts the equality $1 = u = s_2 + em_2$.

Therefore, $u = 0$ and $s_2 = m_2 = 0$. From the first two relations we get $\hat{q} = 7v$. Thus, $\hat{q} = 7$, $v = 1$, $e = 2$, $w = 1$, $m_3 = m_5 = 0$, and $s_3 = s_5 = 1$. By Lemma 4.12 the group $\mathrm{Cl}(\hat{X})$ is torsion free and so $\dim |\Theta| \geq \dim |\bar{S}_5| > 0$. From (vi) of Theorem 1.4 we have $\hat{X} \simeq \mathbb{P}(1^2, 2, 3)$. In particular, $\dim |\Theta| = 1$. Further, similar to (8.2) we have

$$
K_{\bar{X}} + \bar{S}_3 + 2\bar{S}_7 + a_4\bar{E} \sim 0,
$$
$$
\beta_3 + 2\beta_7 = a_4 + \alpha.
$$

This gives us $a_4 = 2\beta_7$ and $s_7 + a_4 = 3$. Since $\dim |S_7| = 2$, $s_7 > 1$, $s_7 = 3$, $\hat{S}_7 \sim 3\Theta$, $a_4 = 0$, and $\beta_7 = 0$, i.e., $P_7 \notin S_7$.

$\square$

Lemma 8.9. (i) $S_2 \cap S_3$ is a reduced irreducible curve.
(ii) $S_2 \cap S_3 \cap S_5 = \{P_7\}$.

Proof. (i) Similar to the proof of (i) of Lemma 7.5.

(ii) Put $C := S_3 \cap S_4$. Assume that $S_2 \cap S_3 \cap S_5 \ni P \neq P_7$. Since $1/7 = S_2 \cdot S_3 \cdot S_5 = S_5 \cdot C$ and $P, P_7 \in S_5 \cap C$, we have $C \subset S_5$. If there is a component $C' \neq C$ of $S_2 \cap S_5$ not contained in S_7, then, as above, $7/15 = S_2 \cdot S_7 \cdot S_7 \geq S_7 \cdot C + S_7 \cdot C' \geq 2/5$, a contradiction. Thus we can write $S_2 \cap S_5 = C + \Gamma$, where Γ is an effective 1-cycle with $\mathrm{Supp}\,\Gamma \subset S_7$. In particular, $P_7 \notin \Gamma$. The divisor $30A$ is Cartier at P_2, P_3, and P_5. We get

$$
\frac{120}{210} = 120A^3 = 30A \cdot S_2 \cdot (S_5 - S_3) = 30A \cdot \Gamma \in \mathbb{Z},
$$

a contradiction. $\square$

Now the proof of (ii) of Theorem 1.4 can be finished similar to 7.7: the pair $(\hat{X}, \hat{S}_3 + \hat{S}_5 + \hat{E} + \hat{S}_7)$ is LC and the corresponding discrepancy of $\bar{S}_2$ is equal to -1.

9. CASE qℚ(X) = 19

Consider the case $q\mathbb{Q}(X) = 19$. By Proposition 3.6 $\mathbf{B} = (3, 4, 5, 7)$. For $r = 3$, 4, 5, 7, let P_r be a (unique) point of index r. In notation of §4, take $\mathscr{M} := |7A| = |S_7|$ and apply Construction (4.5). Near P_5 we have $A \sim -4K_X$ and $\mathscr{M} \sim -3K_X$. By Lemma 4.2 we get $c \leq 1/3$. In particular, the pair $(X, \mathscr{M})$ is not canonical.

Proposition 9.1. *In the above notation, f is the Kawamata blowup of P_5, g is birational, it contracts $\bar{S}_3$, and $\hat{X} \simeq \mathbb{P}(1^2, 2, 3)$. Moreover, $\hat{S}_4 \sim \hat{S}_7 \sim \Theta$, $\hat{E} \sim 3\Theta$, and $\hat{S}_5 \sim 2\Theta$.*

Proof. Similar to (5.1)-(5.2) we have for some $a_1, a_2, a_3, a_4 \in \mathbb{Z}$:

(9.2)
$$
\begin{aligned}
K_{\bar{X}} + 5\bar{S}_3 + \bar{S}_4 + a_1\bar{E} &\sim 0, \\
K_{\bar{X}} + \bar{S}_3 + 4\bar{S}_4 + a_2\bar{E} &\sim 0, \\
K_{\bar{X}} + \bar{S}_4 + 3\bar{S}_5 + a_3\bar{E} &\sim 0, \\
K_{\bar{X}} + \bar{S}_5 + 2\bar{S}_7 + a_4\bar{E} &\sim 0,
\end{aligned}
$$

(9.3)
$$
\begin{aligned}
5\beta_3 + \beta_4 &= a_1 + \alpha, \\
\beta_3 + 4\beta_4 &= a_2 + \alpha, \\
\beta_4 + 3\beta_5 &= a_3 + \alpha, \\
\beta_5 + 2\beta_7 &= a_4 + \alpha.
\end{aligned}
$$

Remark 9.4. Since $S_7 \in \mathscr{M}$ is a general member, by (4.8) we have $c = \alpha/\beta_7 \leq 1/3$, so $3\alpha \leq \beta_7$ and $a_4 \geq 5\alpha + \beta_5$. Further, $S_3 + S_4 \sim S_7$. Thus, $\beta_3 + \beta_4 \geq \beta_7 \geq 3\alpha$, $a_1 \geq 4\beta_3 + 2\alpha$, and $a_2 \geq 3\beta_4 + 2\alpha$.

Assume that $\hat{X}$ is a surface. Then $\hat{X}$ is such as in Lemma 4.13. From the first and second relations in (9.2) we obtain that S_3 and S_4 are g-vertical. Since $\dim |\bar{S}_k| = 0$, $\dim |g(\bar{S}_k)| = 0$, $k = 3$, 4. Hence, $K_{\hat{X}}^2 \leq 6$ and the curves $g(\bar{S}_k)$ are in fact lines on $\hat{X}$. In particular, $g(\bar{S}_3) \sim g(\bar{S}_4)$. This implies $\bar{S}_3 \sim \bar{S}_4$ and $S_3 \sim S_4$, a contradiction.

Now assume that $\hat{X}$ is a curve and let G be a general fiber of g. Clearly, divisors $\bar{S}_3$ and $\bar{S}_4$ are g-vertical. If the divisor $\bar{S}_5$ is also g-vertical, then $k_3\bar{S}_3 \sim k_4\bar{S}_4 \sim k_5\bar{S}_5 \sim G$, where the k_i are the multiplicities of corresponding fibres. Considering proper transforms on X we get $3k_3 = 4k_4 = 5k_5$ and so $k_3 = 20k$, $k_4 = 14k$, $k_5 = 12k$ for some $k \in \mathbb{Z}$. This contradicts the main result of [MP08a]. Therefore the divisor $\bar{S}_5$ is g-horizontal. In this case the degree of the general fiber is 9. As above we have $k_3\bar{S}_3 \sim k_4\bar{S}_4 \sim G$,

$3k_3 = 4k_4$. So, $k_3 = 4k$, $k_4 = 3k$, $k \in \mathbb{Z}$. Again by [MP08a] g has no fibers of multiplicity divisible by 4.

From now on we assume that g is birational. Then

(9.5) $\qquad \hat{q} = 5s_3 + s_4 + a_1 e = s_3 + 4s_4 + a_2 e = s_4 + 3s_5 + a_3 e.$

Consider the case where $f(E)$ is either a curve or a Gorenstein point on X. Then α and β_k are integers. By Remark 9.4

$$a_1 + a_2 = 5(\beta_3 + \beta_4) + \beta_3 - 2\alpha \geq 13\alpha \geq 13.$$

On the other hand, from (9.5) we obtain $2\hat{q} \geq 6s_3 + 5s_4 + 13 \geq 18$. So, $\hat{q} \geq 9$ (both $\bar{S}_3$ and $\bar{S}_4$ cannot be contracted). In this case, the group $\mathrm{Cl}(\hat{X})$ is torsion free and by Lemma 4.12 we have $\hat{E} \geq 3\Theta$. Since $a_4 \geq 5$, we have $\hat{E} \sim 3\Theta$, $\hat{q} \geq 15$, and $\bar{S}_3$ is contracted. In this situation, $|\Theta| = \varnothing$, so s_5, $s_7 \geq 2$. This contradicts the fourth relation in (9.2).

Therefore $P := f(E)$ is a non-Gorenstein point of index $r = 3, 4, 5$ or 7. Similar to 5.4 we have $\alpha = 1/r$ and

r	β_3	β_4	β_5	β_7	a_1	a_2	a_3
3	m_3	$\frac{1}{3} + m_4$	$\frac{2}{3} + m_5$	$\frac{1}{3} + m_7$	$5m_3 + m_4$	$1 + m_3 + 4m_4$	$2 + m_4 + 3m_5$
4	$\frac{1}{4} + m_3$	m_4	$\frac{3}{4} + m_5$	$\frac{1}{4} + m_7$	$1 + 5m_3 + m_4$	$m_3 + 4m_4$	$2 + m_4 + 3m_5$
5	$\frac{2}{5} + m_3$	$\frac{1}{5} + m_4$	m_5	$\frac{3}{5} + m_7$	$2 + 5m_3 + m_4$	$1 + m_3 + 4m_4$	$m_4 + 3m_5$
7	$\frac{2}{7} + m_3$	$\frac{5}{7} + m_4$	$\frac{1}{7} + m_5$	m_7	$2 + 5m_3 + m_4$	$3 + m_3 + 4m_4$	$1 + m_4 + 3m_5$

Claim 9.6. (i) *If $r = 3$ or 4, then $m_7 \geq 1$ and $m_3 + m_4 \geq 1$.*
(ii) *If $r = 7$, then $m_7 \geq 1$.*

Proof. To get inequalities for m_7 we use $1/3 \geq c = \alpha/\beta_7 = 1/r\beta_7$, $r\beta_7 \geq 3$. The inequalities for $m_3 + m_4$ follows from $\beta_3 + \beta_4 \geq \beta_7$. $\square$

Thus, in all cases $a_1, a_2 \geq 1$. Put

$$u := s_3 + em_3, \qquad v := s_4 + em_4, \qquad w := s_5 + em_5.$$

9.7. Case: $r = 3$. Then $u + v > e(m_3 + m_4) \geq e$ by Claim 9.6. Further,

$$
\begin{aligned}
\hat{q} &= 5s_3 + s_4 + (5m_3 + m_4)e &= 5u + v, \\
\hat{q} &= s_3 + 4s_4 + (1 + m_3 + 4m_4)e &= u + 4v + e, \\
\hat{q} &= s_4 + 3s_5 + (2 + m_4 + 3m_5)e &= v + 3w + 2e.
\end{aligned}
$$

If $u = 0$, then $v = \hat{q} = e + 4v$, a contradiction.

Assume that $u \geq 2$. Then $\hat{q} \geq 10$, $u \leq 3$, the group $\mathrm{Cl}(\hat{X})$ is torsion free and by Lemma 4.12 we have $e \geq 3$. If $u = 2$, then $v \geq 2$, $\hat{q} \geq 13$, $v = \hat{q} - 10$, and $e \leq \hat{q} - 2 - 4v \leq 2$, a contradiction. If $u = 3$, then $v = 2$, $e = 6$, $\hat{q} = 17$, and $m_3 = m_4 = 0$. This contradicts Claim 9.6.

Therefore, $u = 1$. Then $v = \hat{q} - 5$, $19 = e + 3\hat{q}$, and $\hat{q} \leq 6$. We get only one solution: $\hat{q} = 6$, $u = v = w = e = 1$. Recall that $m_3 + m_4 \geq 1$ by Claim 9.6. Hence either $s_3 = 0$ and $\hat{S}_4 \sim_{\mathbb{Q}} \hat{S}_5 \sim_{\mathbb{Q}} \hat{E} \sim_{\mathbb{Q}} \Theta$ or $s_4 = 0$ and $\hat{S}_3 \sim_{\mathbb{Q}} \hat{S}_5 \sim_{\mathbb{Q}} \hat{E} \sim_{\mathbb{Q}} \Theta$. In both cases $\hat{S}_5 \not\sim_{\mathbb{Q}} \hat{E}$ (otherwise $\bar{S}_5 \sim \bar{E} + l\bar{F}$ for some $l \in \mathbb{Z}$ and so $S_5 \sim lF$, a contradiction). Then we get a contradiction by Lemma 3.11.

9.8. Case: $r = 4$. As in the previous case, $u + v > e$ and

$$\hat{q} = 5s_3 + s_4 + (1 + 5m_3 + m_4)e = 5u + v + e,$$
$$\hat{q} = s_3 + 4s_4 + (m_3 + 4m_4)e = u + 4v.$$

If u is even, then so is $\hat{q}$. Hence, $\hat{q} \leq 10$. From the first relation we have $u = 0$, $\hat{q} = 4v$, and $e = 3v$. This contradicts $u + v > e$. Therefore u is odd.

Assume that $u = 1$. Then $\hat{q} = 5 + v + e = 1 + 4v$ and $e = 3v - 4$. Since $u + v > e$, there is only one possibility: $v = e = 2$, $\hat{q} = 9$. Then the group $\mathrm{Cl}(\hat{X})$ is torsion free. By Lemma 4.12 we have $F \in |2A| \neq \varnothing$, a contradiction.

Finally, assume $u \geq 3$. Then $u = 3$ and $\hat{q} = 15 + v + e = 3 + 4v \geq 16$. Thus, $\hat{q} = 19$, $v = 4$, and $e = 0$, a contradiction.

9.9. Case: $r = 7$. Then

$$\hat{q} = 5s_3 + s_4 + (2 + 5m_3 + m_4)e = 5u + v + 2e,$$
$$\hat{q} = s_3 + 4s_4 + (3 + m_3 + 4m_4)e = u + 4v + 3e,$$
$$\hat{q} = s_4 + 3s_5 + (1 + m_4 + 3m_5)e = v + 3w + e.$$

In this case $u = (3v + e)/4 > 0$. Assume that $u \geq 2$. Then $\hat{q} \geq 13$ and the group $\mathrm{Cl}(\hat{X})$ is torsion free. By Lemma 4.12 we have $e \geq 3$. Further, $u = 2$, and $\hat{q} \geq 17$. We get $m_3 = 0$, $s_3 = 2$, $e \geq 4$, $\hat{q} = 19$, $e = 4$, and $v = 1$. This contradicts the last relation.

Therefore, $u = 1$. Then $3v + e = 4$. Assume that $e = 4$. Then $v = 0$, $\hat{q} = 13$, $w = 3$, $s_4 = 0$, $s_3 = 1$, and $m_4 = m_3 = 0$. Since $\dim |\Theta| = \dim |2\Theta| = 0$, we have $s_5 \geq 3$. Recall that $m_7 \geq 1$ by Claim 9.6. Hence, $\beta_7 \geq 1$ and $a_4 = 2\beta_7 \geq 2$. This contradicts the fourth relation in (9.2).

Therefore, $e < 4$. In this case, $e = 1$, $v = 1$, and $\hat{q} = 8$. Then $\hat{E} \sim_{\mathbb{Q}} \Theta$ and either $\hat{S}_3 \sim_{\mathbb{Q}} \Theta$ or $\hat{S}_4 \sim_{\mathbb{Q}} \Theta$ (because $u = v = 1$). This contradicts (vi) of Theorem 1.4.

9.10. Case: $r = 5$. From (9.2) we obtain

$$(9.11) \quad \begin{aligned} \hat{q} &= 5s_3 + s_4 + (2 + 5m_3 + m_4)e = 5u + v + 2e, \\ \hat{q} &= s_3 + 4s_4 + (1 + m_3 + 4m_4)e = u + 4v + e, \\ \hat{q} &= s_4 + 3s_5 + (m_4 + 3m_5)e = v + 3w. \end{aligned}$$

Then $e = 3v - 4u$. If $u \geq 2$, then $e = 3v - 4u \leq 3v - 6$, and so $v \geq 3$. Hence, $\hat{q} \geq 15$ and the group $\mathrm{Cl}(\hat{X})$ is torsion free. By Lemma 4.12 we have $e \geq 3$. So $\hat{q} = 19$, $e = 3$, $s_3 = 0$, and $2 = u = em_3 \geq 3$, a contradiction.

Assume that $u = 1$, then $e = 3v - 4$ and $v \geq 2$. Further, $\hat{q} = 7v - 3 = v + 3w \leq 19$. We get $\hat{q} = 11$ and $e = 2$. This contradicts Lemma 4.12.

Therefore, $u = 0$. Then $e = 3v$, $\hat{q} = 7v = 7$, $v = 1$, $e = 3$, and $w = 2$. By Lemma 4.12 the group $\mathrm{Cl}(\hat{X})$ is torsion free. Thus $s_3 = 0$, i.e., $\bar{S}_3$ is contracted, $s_4 = 1$, $s_5 = 2$, and $m_5 = \beta_5 = 0$. This means, in particular, that $P_5 \notin S_5$. From the fourth relation in (9.2) we get $a_4 = 1$ and $s_7 = 1$. In particular, $\dim |\Theta| > 0$ and $\hat{X} \simeq \mathbb{P}(1^2, 2, 3)$ by (vi) of Theorem 1.4.

$\square$

Lemma 9.12. (i) $S_3 \cap S_4$ *is a reduced irreducible curve.*
  (ii) $S_3 \cap S_4 \cap S_7 = \{P_5\}$.

Proof. (i) Similar to the proof of (i) of Lemma 7.5.

(ii) Put $C := S_3 \cap S_4$. Assume that $S_3 \cap S_4 \cap S_7 \ni P \neq P_5$. Since $1/5 = S_3 \cdot S_4 \cdot S_7 = S_7 \cdot C$ and $P, P_5 \in S_7 \cap C$, we have $C \subset S_7$. If there is a component $C' \neq C$ of $S_3 \cap S_7$ not contained in S_5, then, as above, $1/4 = S_3 \cdot S_7 \cdot S_5 \geq S_5 \cdot C + S_5 \cdot C' \geq 2/7$, a contradiction. Thus we can write $S_3 \cap S_7 = C + \Gamma$, where Γ is an effective 1-cycle with $\mathrm{Supp}\,\Gamma \subset S_5$. In particular, $P_5 \notin S_5$. The divisor $84A$ is Cartier at P_3, P_4, and P_7. We get

$$\frac{9}{5} = 84A \cdot S_3 \cdot (S_7 - S_4) = 84A \cdot \Gamma \in \mathbb{Z},$$

a contradiction. $\square$

Now the proof of (i) of Theorem 1.4 can be finished similar to 7.7.

10. Toric Sarkisov links

Proposition 10.1. *Let X be a toric $\mathbb{Q}$-Fano threefold and let $P \in X$ be a cyclic quotient singularity of index r. Let $f\colon \tilde{X} \to X$ be the Kawamata blowup of $P \in X$. Then a general member of $|-K_X|$ is a normal surface having at worst Du Val singularities. The linear system $|-K_X|$ has only isolated base points. In particular, $-K_{\tilde{X}}$ is nef and big. The map $f\colon \tilde{X} \to X$ can be completed by a toric Sarkisov link (cf. (4.5)).*

Proof. This can be shown by explicit computations in all cases of Proposition 1.3. Consider, for example, the case $X = \mathbb{P}(3, 4, 5, 7)$. Let x_3, x_4, x_5, x_7 be quasi-homogeneous coordinates in $\mathbb{P}(3, 4, 5, 7)$. A section $S \in |-K_X|$ is given by a quasi-homogeneous polynomial of degree 19. By taking this polynomial as a general linear combination of $x_3^5 x_4$, $x_3^3 x_5^2$, $x_3^4 x_7$, $x_4 x_5^3$, $x_4^3 x_7$, $x_5 x_7^2$ we see that the base locus of $|-K_X|$ is the union of four coordinate points and the surface S has only quotient singularities.

Since K_S is Cartier, the singularities of S are Du Val. Further, we can write $K_{\tilde{X}} + \tilde{S} = f^*(K_X + S) \sim 0$, where $\tilde{S}$ is the proper transform of S. Hence, $\tilde{S} \in |-K_{\tilde{S}}|$ and the linear system $|-K_{\tilde{X}}|$ has only isolated base points outside of $f^{-1}(P)$. In particular, $-K_{\tilde{X}}$ is nef. It is easy to check that $-K_{\tilde{X}}^3 > 0$, i.e., $-K_{\tilde{X}}$ is big. Recall that $\rho(\tilde{X}) = 2$. So, the Mori cone $\mathrm{NE}(\tilde{X})$ has exactly two extremal rays, say R_1 and R_2. Let R_1 is generated by f-exceptional curves. If $-K_{\tilde{X}}$ is ample, we run the MMP starting from R_2. Otherwise we make a flop in R_2 and run the MMP. Clearly, we obtain Sarkisov link (4.5). $\qquad\square$

Explicitly, for weighted projective spaces from Proposition 1.3, we have the following diagram of Sarkisov links. Here an arrow $X_1 \xrightarrow{\frac{1}{r}} X_2$ indicates that there is Sarkisov link described above that starts from Kawamata blowup of a cyclic quotient singularity of index $r > 1$ on X_1 and the target variety is X_2.

REFERENCES
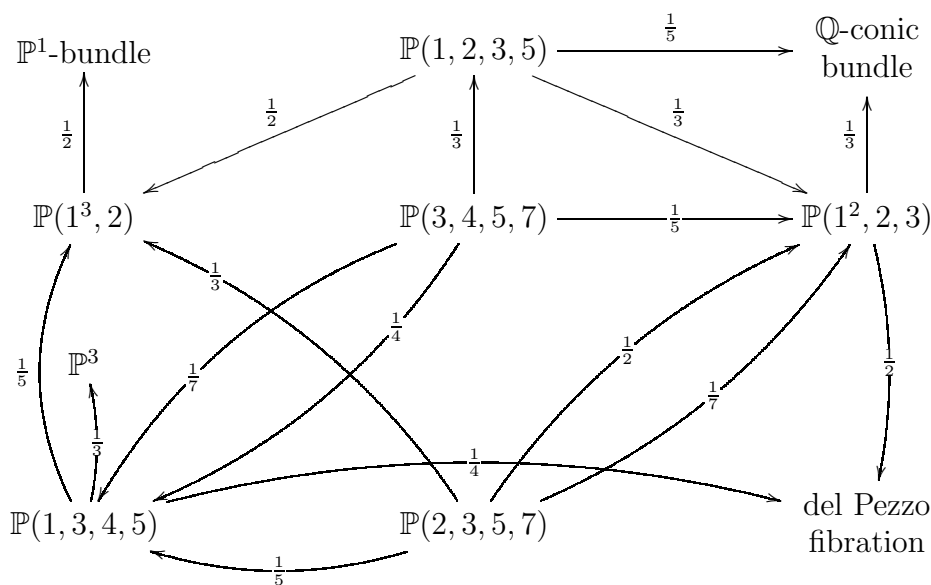
[Ale94]   V. Alexeev. General elephants of **Q**-Fano 3-folds. *Compositio Math.*, 91(1):91–116, 1994.

[BB92]   A. A. Borisov and L. A. Borisov. Singular toric Fano three-folds. *Mat. Sb.*, 183(2):134–141, 1992.

[Fuj75]   T. Fujita. On the structure of polarized varieties with Δ-genera zero. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 22:103–115, 1975.

[IP99]   V. A. Iskovskikh and Y. G. Prokhorov. *Fano varieties. Algebraic geometry. V.*, volume 47 of *Encyclopaedia Math. Sci.* Springer, Berlin, 1999.

[Kaw92a]   Y. Kawamata. Boundedness of $\mathbb{Q}$-Fano threefolds. In *Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989)*, volume 131 of *Contemp. Math.*, pages 439–445, Providence, RI, 1992. Amer. Math. Soc.

[Kaw92b]   Y. Kawamata. The minimal discrepancy coefficients of terminal singularities in dimension three. Appendix to V.V. Shokurov's paper "3-fold log flips". *Russ. Acad. Sci., Izv., Math.*, 40(1):95–202, 1992.

[Kaw96]   Y. Kawamata. Divisorial contractions to 3-dimensional terminal quotient singularities. In *Higher-dimensional complex varieties (Trento, 1994)*, pages 241–246. de Gruyter, Berlin, 1996.

[KMMT00] J. Kollár, Y. Miyaoka, S. Mori, and H. Takagi. Boundedness of canonical $\mathbb{Q}$-Fano 3-folds. *Proc. Japan Acad. Ser. A Math. Sci.*, 76(5):73–77, 2000.

[KO73]   S. Kobayashi and T. Ochiai. Characterizations of complex projective spaces and hyperquadrics. *J. Math. Kyoto Univ.*, 13:31–47, 1973.

[Kol92]   J. Kollár, editor. *Flips and abundance for algebraic threefolds.* Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[McK01]   J. McKernan. A simple characterization of toric varieties. In *Proc. Algebraic Geom. Symp. Kinosaki*, pages 59–72, 2001.

[Mor85a]   S. Mori. On 3-dimensional terminal singularities. *Nagoya Math. J.*, 98:43–66, 1985.

[Mor85b]   D. R. Morrison. The birational geometry of surfaces with rational double points. *Math. Ann.*, 271(3):415–438, 1985.

[MP08a]   S. Mori and Y. Prokhorov. Multiple fibers of del Pezzo fibrations. e-print arXiv:0806.3809, 2008.

[MP08b]   S. Mori and Y. Prokhorov. On **Q**-conic bundles. *Publ. RIMS*, 44(2):315–369, 2008.

[MP08c]   S. Mori and Y. Prokhorov. On **Q**-conic bundles, II. *Publ. RIMS*, 44(3):955–971, 2008.

[MZ88]   M. Miyanishi and D.-Q. Zhang. Gorenstein log del Pezzo surfaces of rank one. *J. Algebra*, 118(1):63–84, 1988.

[Pro05]   Y. G. Prokhorov. On the degree of Fano threefolds with canonical Gorenstein singularities. *Russian Acad. Sci. Sb. Math.*, 196(1):81–122, 2005.

[Pro07]   Y. G. Prokhorov. The degree of **Q**-Fano threefolds. *Russian Acad. Sci. Sb. Math.*, 198(11):1683–1702, 2007.

[Rei87]   M. Reid. Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.

[San96]   T. Sano. Classification of non-Gorenstein **Q**-Fano d-folds of Fano index greater than $d - 2$. *Nagoya Math. J.*, 142:133–143, 1996.

[Sho85]   V. V. Shokurov. A nonvanishing theorem. *Izv. Akad. Nauk SSSR Ser. Mat.*, 49(3):635–651, 1985.

[Sho92]   V. V. Shokurov. Three-dimensional log perestroikas. *Izv. Ross. Akad. Nauk Ser. Mat.*, 56(1):105–203, 1992.

[Suz04]   K. Suzuki. On Fano indices of $\mathbb{Q}$-Fano 3-folds. *Manuscripta Math.*, 114(2):229–246, 2004.

[PARI]   The PARI Group, Bordeaux. *PARI/GP, version* 2.3.4, 2008. available from
         `http://pari.math.u-bordeaux.fr/`.

DEPARTMENT OF ALGEBRA, FACULTY OF MATHEMATICS, MOSCOW STATE UNI-
VERSITY, MOSCOW 117234, RUSSIA
   E-mail address: `prokhoro@gmail.com`